# Moonshine paths for $3A$ and $6A$ nodes of the extended $E_8$-diagram

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**Abstract**

We continue the program, begun in [13], to make a moonshine path between a node of the extended $E_8$-diagram and the Monster. Our theory is a concrete model expressing some of the mysterious connections identified by John McKay, George Glauberman and Simon Norton. In this article, we treat the $3A$ and $6A$-nodes. We determine the orbits of triples $(x, y, z)$ in the Monster where $z \in 2B$, $x, y \in 2A \cap C(z)$ and $xy \in 3A \cup 6A$. Such $x, y$ correspond to a rootless $EE_8$-pair in the Leech lattice. For the $3A$ and $6A$ cases, we shall say something about the "half Weyl groups", which are proposed in the Glauberman-Norton theory. Most work in this article is with lattices, due to their connection with dihedral subgroups of the Monster. These lattices are $M+N$, where $M, N$ is the relevant pair of $EE_8$-sublattices, and their annihilators in the Leech lattice. The isometry groups of these four lattices are analyzed.

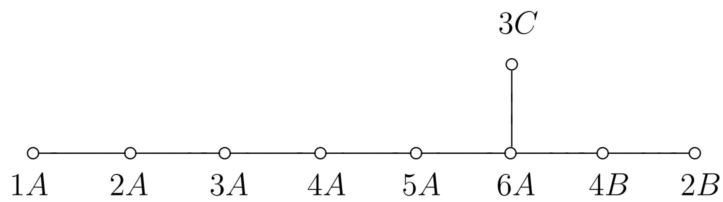
# Contents



# 1 Introduction

Moonshine path theory [13] is intended to understand the discoveries of McKay [20] and Glauberman-Norton [7] which connect the extended $E_8$-diagram and the Monster, denoted $\mathbb{M}$, and place these relationships in a broader mathematical context. The paths involve series of small steps, each using established mathematical theories. The introduction of [13] has a detailed discussion of context, which involves lattices, vertex operator algebras (VOAs), Lie theory and finite groups. In [13] and [14], we treated the cases of the $3C$-node and the $5A$-node in detail. The present article treats the cases of the $3A$-node and the $6A$-node.

Let us first review the background and the main ideas in [13]. It is well known (cf. [1]) that $2A$-involutions of the Monster simple group $\mathbb{M}$ satisfy a 6-transposition property, that is, given a pair of $2A$-involutions $(x, y)$ in $\mathbb{M}$, the product $xy$ has order $\leq 6$. John McKay [20] noticed a remarkable correspondence with the extended $E_8$-diagram $\tilde{E}_8$ as follows.




(1)

There are 9 conjugacy classes of such pairs $(x, y)$, and the orders of the 8 products $|xy|$, for $x \neq y$, are the coefficients of the highest root in the $E_8$-root system. The 9 nodes are labeled with 9 conjugacy classes of $\mathbb{M}$ containing the $xy$.

In 2001, George Glauberman and Simon Norton [7] enriched this theory by adding details about the centralizers in the Monster of such pairs of involutions and relations involving the associated modular forms. Let $(x, y)$ be such a pair and let $n(x, y)$ be its associated node. Let $n'(x, y)$ be the subgraph of $\tilde{E}_8$ which is supported at the set of nodes complementary to $\{n(x, y)\}$. If $z$ is a certain $2B$ involution which commutes with $\langle x, y\rangle$, Glauberman and Norton give a lot of detail about $C(x, y, z)$. In particular, they proposed that $C(x, y, z)$ has a “new” relation to the extended $E_8$-diagram, namely that $C(x, y, z)/O_2(C(x, y, z))$ looks roughly like “half” of the Weyl group corresponding to the subdiagram $n'(x, y)$. This article shows that this new relation is not valid for the $6A$-case. See our main theorems. The relations for $3C$ and $5A$-cases are valid [13, 14].

## 1.1 About the proof

The main idea of [13] is to transfer a problem in group theory to a study of certain subVOAs of the Moonshine VOA $V^\natural$ and some lattices of the Leech lattice. Thus, our the articles on moonshine paths involve a mixture of techniques, finite group theory, internal analysis of lattices spanned by rootless $EE_8$-pairs and analysis of sublattices of the Leech lattice. The bijection between $2A$ involutions of $\mathbb{M}$ and conformal vectors of central charge $1/2$ (abbreviated as cvcc$\frac{1}{2}$ ) in the Moonshine VOA $V^\natural$ is foundational. See the

theory of Miyamoto involutions [21].

The first observation is that the dihedral group $\langle x, y\rangle$ is uniquely determined by the subVOA generated by the associated cvcc$\frac{1}{2}$ $e'$ and $f'$ [1, 19]. We noticed that the subdiagram $n'(x, y)$ defines an automorphism $r = r(x, y)$ of exponential type in $Aut(V_{E_8})$ and one can construct a pair of conformal vectors $e$ and $f$ of central charge $1/2$ in a lattice VOA $V_{EE_8}$ by using $r$. We also explained in [13] that $r(x, y)$ is conjugate in $Aut(V_{E_8})$ to an automorphism $\hat{h}(x, y)$ in a torus normalizer in $Aut(V_{E_8}) \cong E_8(\mathbb{C})$. This approach led us to consider a pair of $EE_8$-sublattices $M$ and $N$ in $E_8 \perp E_8$. We showed that the pair $(M, N)$ can be isometrically embedded into the Leech lattice $\Lambda$ and that the subVOA $\langle e, f\rangle$ of $V_{EE_8}$ generated by $e$ and $f$ can be embedded into the VOA $V_\Lambda^+ \subset V^\natural$. Many properties of the dihedral group $\langle \tau_e, \tau_f\rangle$ generated by the Miyamoto involutions can be studied by examining embeddings of the pair $(M, N)$ in $\Lambda$. In particular, the centralizer $C(\tau_e, \tau_f, z)$ has a factor subgroup which looks like the common stabilizer of $M$ and $N$ in $O(\Lambda)/\{\pm 1\}$ (see Corollary 3.5 and 4.5).

## 1.2 Statements of main results.

**Main Theorem 1** (Proposition 3.3 and Theorem 3.25)**.** *Let $x, y, z \in \mathbb{M}$ such that $x, y \in 2A, xy \in 3A$, and $z \in 2B \cap C_{\mathbb{M}}(x, y)$. Then the triple $(x, y, z)$ is unique up to conjugation by $\mathbb{M}$. Moreover, there exist an $EE_8$ pair $(M, N)$ in $\Lambda$ such that $M + N \cong DIH_6(14)$ and the centralizer $C_{\mathbb{M}}(x, y, z)$ has a homomorphism onto $C_{O(\Lambda)}(\langle t_M, t_N\rangle)/O_2(C_{O(\Lambda)}(\langle t_M, t_N\rangle))$, which is isomorphic to "half" of the Weyl group of type $A_2 + E_6$.*

**Main Theorem 2** (Theorem 4.2 and 4.4)**.** *Consider triples $(x, y, z)$ so that $x, y, z \in \mathbb{M}$ $x, y \in 2A, xy \in 6A$, and $z \in 2B \cap C_{\mathbb{M}}(x, y)$. There are two orbits on the set of such triples $(x, y, z)$ under conjugacy by $\mathbb{M}$.*

***Orbit*** $6A.1$***:*** *$(xy)^3 \in O_2(C_{\mathbb{M}}(z))$, and the triple $(x, y, z)$ is conjugate to $(\tau_{e_M}, \tau_{\varphi_{\alpha/2}(e_N)}, z)$, where $M$ and $N$ are $EE_8$-sublattices of $\Lambda$, $M + N = DIH_6(14)$ and $\alpha \in M \cap N(4)$.*

***Orbit*** $6A.2$***:*** *$(xy)^3 \notin O_2(C_{\mathbb{M}}(z))$, and the triple $(x, y, z)$ is conjugate to $(\tau_{e_M}, \tau_{e_N}, z)$, where $M$ and $N$ are $EE_8$-sublattices of $\Lambda$ and $M + N = DIH_{12}(16)$.*

In all cases, the centralizer of $\langle x, y, z\rangle$ is determined.

**Main Theorem 3** (Theorem 4.8)**.** *Suppose that the triple* $(x, y, z)$ *is in the orbit* $6A.1$*. Then* $C_{\mathbb{M}}(x, y, z)$ *has a homomorphism onto*

$$C_{Stab_{O(\Lambda)}(\mathbb{Z}\alpha)}(t_M, t_N)/O_2(C_{Stab_{O(\Lambda)}(\mathbb{Z}\alpha)}(t_M, t_N)) \cong PSU(4, 2).2.$$

*The kernel* $K$ *is a* 2*-group of order* $2^{11}$ *and we have an exact sequence*

$$1 \to \langle z \rangle \to K \to \{\varphi_\beta \mid \beta \in \Lambda \text{ and } \langle \beta, M + N \rangle \in 2\mathbb{Z}\} \to 1.$$

**Main Theorem 4** (Theorem 4.15)**.** *Suppose that the triple* $(x, y, z)$ *is in the orbit* $6A.2$*. Then the natural map* $\theta$ *of* $C_{\mathbb{M}}(x, y, z)$ *to* $C_{\mathbb{M}}(z)/O_2(C_{\mathbb{M}}(z))$ *has image isomorphic to*

$$C_{O(\Lambda)}(t_M, t_N)/\langle \pm 1 \rangle \cong (3 \times 2.(Alt_4 \times Alt_4).2).2.$$

*The kernel* $\tilde{K}$ *of* $\theta$ *is a group of order* $2^9$ *and the sequence*

$$1 \to \langle z \rangle \to \tilde{K} \to \{\varphi_\beta \mid \beta \in \Lambda \text{ and } \langle \beta, M + N \rangle \in 2\mathbb{Z}\} \to 1$$

*is exact. (For a description of the group* $C_{O(\Lambda)}(t_M, t_N)$ *as an index* 2 *subgroup of* $2.(Dih_6 \times O^+(4, 3))$*, see (4.15) and (B.9).*

We also show that $O(\Lambda)$ has one orbit on ordered pairs $(M, N)$ of $EE_8$-sublattices so that $M+N$ has type $DIH_6(14)$ (3.19) and one orbit on ordered pairs $(M, N)$ of $EE_8$-sublattices so that $M + N$ has type $DIH_{12}(16)$ (4.11). There are analogous transitivity results in [13], [14].

A striking feature of the Glauberman-Norton theory is that the stabilizer of a triple $(x, y, z)$ (modulo $O_2$) seemed to be roughly "half" the Weyl group of the corresponding node of the extended $E_8$-diagram. Our results so far confirm this for several nodes, [13, 14] but this is not the case for the $6A$-node, for either of the two orbits, 6A.1 or 6A.2.

The Weyl group associated to removal of the $6A$-node has shape

$$Weyl(A_5) \times Weyl(A_2) \times Weyl(A_1) \cong Sym_5 \times Sym_3 \times Sym_2.$$

The quotients $C_{\mathbb{M}}(x, y.z)/O_2(C_{\mathbb{M}}(x, y, z))$ are described in Main Theorems 3 and 4. Neither can be interpreted as half the above Weyl group.

Our moonshine path theories for the $3C$, $5A$, $3A$ and $6A$ cases have different degrees of confirmation of the Glauberman-Norton observations. Aspects of the $3C$ path [13] are especially nice.

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## Notation and Terminology

| Notation | Explanation | Examples in text |
|---|---|---|
| $2A, 2B, 3A, \ldots$ | conjugacy classes of the Monster: the first number denotes the order of the elements and the second letter is arranged in descending order of the size of the centralizers | Introduction |
| $A_1, \cdots, E_8$ | root lattice for root system $\Phi_{A_1}, \ldots, \Phi_{E_8}$ | (2.3) |
| $AA_1, \cdots,$ $EE_8$ | lattice isometric to $\sqrt{2}$ times the lattice $A_1, \cdots, E_8$ | (2.3) |
| $A \circ B$ | central product of groups $A$ and $B$ | (3.27) |
| cvcc$\frac{1}{2}$ | conformal vector of central charge 1/2 | (3.3) |
| $\mathcal{D}(L)$ | discriminant group of integral lattice $L$: $\mathcal{D}(L) = L^*/L$ | (3.6) (4.9) |
| $DIH_n(k)$ | the sum of a $EE_8$ pair $(M, N)$ such that the corresponding SSD involutions generate a dihedral of order $n$ and $M + N$ has rank $k$ | (3.6) |
| $E$ | a particular $EE_8$-sublattice | (4.12) |
| $EE_8$-involution | SSD involution whose negated space is isometric to $EE_8$ | (3.6) |
| $e_M$ | a cvcc$\frac{1}{2}$ associated to an $EE_8$ sublattice $M$ | (2.4) |
| $G$ | complex isometry group of the Coxeter-Todd lattice $K_{12}$ | (3.8) |
| $g_1$ | an order 3 isometry in $O_3(O(K_{12}))$ | (3.6) |
| $H$ | the subgroup generated by complex refections defined by norm 4 vectors of $K_{12}$ | (3.10) |

| Notation | Explanation | Examples in text |
|---|---|---|
| $J$ | a sublattice of $\Lambda$ isometric to $K_{12}$ | (3.6) |
| $\mathcal{H}$ | the hexacode | (3.11) |
| $K$ | a sublattice of $\Lambda$ isometric to $K_{12}$ | (3.6) |
| $K_{12}$ | the Coxeter-Todd lattice of rank 12 | (3.7) |
| $\Lambda$ | the Leech lattice, rank 24 | |
| $\mathbb{M}$ | Monster sporadic group | Introduction |
| $\mu$ | the natural surjection from $C_{\mathbb{M}}(z) \to Aut(V_\Lambda^+) \cong 2^{24}.Co_1$ | (2.8) |
| $O(X)$ | the isometry group of the quadratic space $X$ | (3.6), (3.13) |
| $O_p(G)$ | largest normal $p$-subgroup of a finite group $G$ ($p$ is a prime) | Introduction |
| $O^\varepsilon(2n,q)$ | orthogonal group of dimension $2n$ over $\mathbb{F}_q$ of type $\varepsilon = \pm$ | (4.14) |
| $P_N$ | the orthogonal projection from a lattice $L$ to $\mathbb{Q} \otimes N^*$ | (2.14) (D.2) |
| RSSD | relatively semi self dual | (2.2) |
| SSD | semi self dual | (2.2) |
| $t_M$ | SSD involution associated to $M$ | (3.6) |
| $\varphi_\alpha(u \otimes e^\beta)$ | a cvcc$\frac{1}{2}$ | (2.7) |
| $\tau_e$ | the Miyamoto involution defined by a cvcc$\frac{1}{2}$ $e$ | (2.5) (2.10) |
| $\xi$ | the natural surjection from $Aut(V_\Lambda^+) \to O(\Lambda)/\langle \pm 1\rangle$ | (2.8) |
| $\omega^\pm(\alpha)$ | a cvcc$\frac{1}{2}$ associated to a norm 4 vector | (2.9) |
| $z$ | an automorphism of $V^\natural$ such that $z\|_{V_\Lambda^+} = 1$ and $z\|_{V_\Lambda^{T,+}} = -1$ | (2.8) |

# 2 Preliminary

We review terminology about rational lattices and involutions. For background, see [11].

**Definition 2.1.** *Let $X$ be a subset of Euclidean space. Define $t_X$ to be the orthogonal transformation which is $-1$ on $X$ and is $1$ on $X^\perp$.*

**Definition 2.2.** A sublattice $M$ of an integral lattice $L$ is *RSSD (relatively semiselfdual)* if and only if $2L \leq M + ann_L(M)$. This implies that $t_M$ maps $L$ to $L$ and is equivalent to this property when $M$ is a direct summand.

The property that $2M^* \leq M$ is called *SSD (semiselfdual)*. It implies the RSSD property, but the RSSD property is often more useful.

**Notation 2.3.** *We use $XX_n$ to denote the lattice which is isometric to $\sqrt{2}$ times the root lattice of type $X_n$. For example, $EE_8$ is the $\sqrt{2}$ times of the root lattice $E_8$.*

## 2.1 Conformal vectors and the Monster

Next we shall recall some facts about conformal vectors of central charge 1/2 (cvcc$\frac{1}{2}$ ) in the lattice type VOA $V_\Lambda^+$ and the Moonshine VOA $V^\natural$.

We use the standard notation for the lattice vertex operator algebra

$$V_L = M(1) \otimes \mathbb{C}\{L\} \tag{2}$$

associated with a positive definite even lattice $L$ [5]. In particular, $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ is an abelian Lie algebra and we extend the bilinear form to $\mathfrak{h}$ by $\mathbb{C}$-linearity. Also, $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k$ is the corresponding affine algebra and $\mathbb{C}k$ is the 1-dimensional center of $\hat{\mathfrak{h}}$. The subspace $M(1) = \mathbb{C}[\alpha_i(n)|1 \leq i \leq d, n < 0]$ for a basis $\{\alpha_1, \ldots, \alpha_d\}$ of $\mathfrak{h}$, where $\alpha(n) = \alpha \otimes t^n$, is the unique irreducible $\hat{\mathfrak{h}}$-module such that $\alpha(n) \cdot 1 = 0$ for all $\alpha \in \mathfrak{h}$ and $n$ nonnegative, and $k$ acts as the scalar 1. Also, $\mathbb{C}\{L\} = span\{e^\beta \mid \beta \in L\}$ is the twisted group algebra of the additive group $L$ such that $e^\beta e^\alpha = (-1)^{\langle \alpha, \beta \rangle} e^\alpha e^\beta$ for any $\alpha, \beta \in L$. The vacuum vector $\mathbf{1}$ of $V_L$ is $1 \otimes e^0$ and the Virasoro element $\omega$ is $\frac{1}{2} \sum_{i=1}^d \beta_i(-1)^2 \cdot \mathbf{1}$ where $\{\beta_1, ..., \beta_d\}$ is an orthonormal basis of $\mathfrak{h}$. For the explicit definition of the corresponding vertex operators, we shall refer to [5] for details.

**Notation 2.4.** *Let $M \cong EE_8$. Define*

$$e_M = \frac{1}{16}\omega_M + \frac{1}{32} \sum_{\alpha \in M(4)} e^\alpha, \tag{3}$$

*where $\omega_M$ is the Virasoro element of $V_M$ and $M(4) = \{\alpha \in M | \langle \alpha, \alpha \rangle = 4\}$. It is shown in [6] that $e_M$ is a simple conformal vector of central charge* $1/2$.

*For $x \in M^*$, define a $\mathbb{Z}$-linear map*

$$\begin{aligned} \langle x, \cdot \rangle : M \to \ & \mathbb{Z}_2 \\ y \mapsto \ & \langle x, y \rangle \mod 2. \end{aligned}$$

*Clearly the map*

$$\varphi : M^* \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}_2)$$
$$x \longmapsto \langle x, \cdot \rangle$$

*is a group homomorphism and* $Ker\varphi = 2M^* = M$. *For any* $x \in M^* = \frac{1}{2}M$, $\langle x, \cdot \rangle$ *induces an automorphism* $\varphi_x$ *of* $V_M$ *given by*

$$\varphi_x(u \otimes e^\alpha) = (-1)^{\langle x, \alpha \rangle} u \otimes e^\alpha \quad \text{for } u \in M(1) \text{ and } \alpha \in M. \tag{4}$$

*Note that* $\varphi_x(e_M)$ *is also a simple conformal vectors of central charge* $1/2$.

**Remark 2.5.** *Given a simple conformal vector* $e$ *of central charge* $1/2$, *one can define an involutive automorphism* $\tau_e \in Aut(V)$, *called Miyamoto involution. If* $V = V^\natural$ *is the Moonshine VOA, then* $\tau_e$ *defines a* $2A$ *involution in the Monster [21].*

**Notation 2.6.** *Let* $\Lambda$ *be the Leech lattice and* $V_\Lambda$ *the lattice VOA associated with* $\Lambda$. *Let* $\theta$ *be a lift of the* $-1$*-isometry of* $\Lambda$ *to* $Aut(V_\Lambda)$. *We use* $V_\Lambda^T$ *to denote the unique* $\theta$*-twisted module of* $V_\Lambda$ *and* $V_\Lambda^+$, $V_\Lambda^{T,+}$ *to denote the fixed point subspaces of* $\theta$ *in* $V_\Lambda$ *and* $V_\Lambda^T$, *respectively. [5].*

**Notation 2.7.** *Let* $\Lambda$ *be the Leech lattice. For each* $\alpha \in L$, *we define* $\varphi_\alpha \in Aut(V_\Lambda)$ *by*

$$\varphi_\alpha(u \otimes e^\beta) = (-1)^{\langle \alpha, \beta \rangle} u \otimes e^\beta \quad \text{for } u \in M(1) \text{ and } \beta \in \Lambda.$$

*Note that* $\varphi_\alpha = \varphi_{\alpha'}$ *if and only if* $\alpha - \alpha' \in 2\Lambda$. *In addition,* $\varphi_\alpha$ *commutes with* $\theta$ *and thus* $\varphi_\alpha$ *also defines an automorphism on* $V_\Lambda^+$.

**Notation 2.8.** *Let* $\theta$, $V_\Lambda^+$, *and* $V_\Lambda^{T,+}$ *be defined as in (2.6). Then the Moonshine VOA* $V^\natural$ *[5] is constructed as a* $\mathbb{Z}_2$*-orbifold of the Leech lattice VOA* $V_\Lambda$, *i.e.,*

$$V^\natural = V_\Lambda^+ \oplus V_\Lambda^{T,+}.$$

*Let* $z$ *be an involution of* $V^\natural$ *acting as* $1$ *and* $-1$ *on* $V_\Lambda^+$ *and* $V_\Lambda^{T,+}$ *respectively. Then* $z$ *defines a involution on* $V^\natural$, *which is in conjugacy class* $2B$ *[5, 21]. Recall from [18] that*

$$C_{Aut(V^\natural)}(z)/\langle z \rangle \cong C_{Aut(V_\Lambda)}(\theta)/\langle \theta \rangle \cong Aut(V_\Lambda^+)$$

*and the sequences*

$$1 \longrightarrow \langle z \rangle \longrightarrow C_{Aut(V^\natural)}(z) \xrightarrow{\mu} Aut(V_\Lambda^+) \to 1$$

*and*

$$1 \longrightarrow \mathrm{Hom}(\Lambda, \mathbb{Z}_2) \longrightarrow Aut(V_\Lambda^+) \ \xrightarrow{\xi}\ O(\Lambda)/\langle \pm 1 \rangle \longrightarrow 1.$$

*are exact.*

The following results can also be found in [18].

**Theorem 2.9** (Theorem 5.18 of [18])**.** *Let $\alpha \in \Lambda(4)$ and let*

$$e := \omega^+(\alpha) = \frac{1}{16}\alpha(-1)^2 \cdot 1 + \frac{1}{4}(e^\alpha + e^{-\alpha}).$$

*Then there is an exact sequence*

$$1 \to K \to C_{Aut(V^\natural)}(z, \tau_e)/\langle z \rangle \to Stab_{O(\Lambda)}(\mathbb{Z}\alpha) \cong Co_2 \to 1,$$

*where $K \cong \{\varphi_\beta \in Hom(\Lambda, \mathbb{Z}_2) \mid \langle \alpha, \beta \rangle \in 2\mathbb{Z}\}$.*

**Remark 2.10.** *Recall from [18] that for any $\alpha \in \Lambda(4)$ and*

$$e = \omega^\pm(\alpha) = \frac{1}{16}\alpha(-1)^2 \cdot 1 \pm \frac{1}{4}(e^\alpha + e^{-\alpha}),$$

*the Miyamoto involution $\tau_e$ acts on $V_\Lambda^+$ as $\varphi_\alpha$ defined in (2.7).*

**Lemma 2.11.** *Let $M$ be a sublattice of $\Lambda$ isomorphic to $EE_8$. Then the sequence*

$$1 \longrightarrow X \longrightarrow Stab_{Aut(V_\Lambda^+)}(V_M^+) \xrightarrow{\xi} C_{O(\Lambda)}(t_M)/\langle \pm 1 \rangle \longrightarrow 1$$

*is exact, where $X = \{\varphi_\alpha \in Hom(\Lambda, \mathbb{Z}_2) \mid \langle \alpha, M \rangle \in 2\mathbb{Z}\}$.*

**Theorem 2.12.** *Let $M$ be a sublattice of $\Lambda$ isomorphic to $EE_8$. Let $e = \varphi_x(e_M)$ be defined as in Notation (2.4). Then the centralizer $C_{AutV^\natural}(\tau_e, z)$ stabilizes the subVOA $V_M^+$ and has the structure $2^{2+8+16}.\Omega^+(8,2)$. Moreover, the map*

$$\xi \circ \mu : C_{AutV^\natural}(\tau_e, z) \to C_{O(\Lambda)}(t_M)/\langle \pm 1 \rangle$$

*is surjective and $C_{AutV^\natural}(\tau_e, z)$ acts on $V_M^+$ as $\Omega^+(8,2)$, which is the quotient of the commutator subgroup of the Weyl group of $E_8$ by its center.*

The following result can be found in [12, Appendix F] (see also [13, 14]).

**Theorem 2.13.** *Except for $DIH_4(15)$, every $EE_8$ pair $(M, N)$ in Table 1 of [12] can be embedded into $\Lambda$.*

**Definition 2.14.** *Let $L$ be an integral lattice and $N$ a sublattice. We denote the orthogonal projection of $L$ to $\mathbb{Q} \otimes N^*$ by $P_N$.*

# 3 $3A$-triples

In this section, we consider a triple of elements $x, y, z \in \mathbb{M}$ such that $x, y \in 2A, xy \in 3A$ and $z \in 2B \cap C_{\mathbb{M}}(x, y)$. We shall show that there is only one orbit of such triples under the action of the Monster and determine their centralizer in $\mathbb{M}$.

**Lemma 3.1.** *Let $x, y, z \in \mathbb{M}$ be such that $x, y \in 2A, xy \in 3A$ and $z \in 2B \cap C_{\mathbb{M}}(x, y)$. Then, there exists a pair of $EE_8$-sublattices $(M, N)$ of $\Lambda$ and $a \in M^*$, $b \in N^*$ such that $M + N \cong DIH_6(14)$ and the triple $(x, y, z)$ is conjugate to $(\tau_{\varphi_a(e_M)}, \tau_{\varphi_b(e_N)}, z)$ in $\mathbb{M}$.*

**Proof.** Up to conjugation, we may assume $z$ acts as 1 on $V_\Lambda^+$ and as $-1$ on $V_\Lambda^{T,+}$ and $(V^\natural)^z = V_\Lambda^+$.

By the 1-1 correspondence between cvcc$\frac{1}{2}$ in $V^\natural$ and 2A involutions of $\mathbb{M}$ [15, 21], we have $x = \tau_e$ and $y = \tau_f$ for some cvcc$\frac{1}{2}$ $e, f \in V^\natural$. Since $z$ centralizes $x$ and $y$, we have $\tau_{ze} = z\tau_e z^{-1} = \tau_e$ and $\tau_{zf} = z\tau_f z^{-1} = \tau_f$. Hence $e$ and $f$ are fixed by $z$ by the 1-1 correspondence.

Since $\tau_e\tau_f$ has order 3, both $e$ and $f$ must be of $EE_8$-type. That means there exists $EE_8$-sublattices $M$ and $N$ and $a \in M^*$ and $b \in N^*$ such that $e = \varphi_a(e_M)$ and $f = \varphi_b(e_N)$ (see (2.4)).

Recall that there are two types of cvcc$\frac{1}{2}$ in $V_\Lambda^+$. If $e = \omega^\pm(\alpha)$ is of $AA_1$-type, then $\tau_e = \varphi_\alpha$ on $V_\Lambda^+$ and $\tau_e \in O_2(Aut(V_\Lambda^+)) = \{\varphi_\alpha \mid \alpha \in \Lambda\} = \mathrm{Hom}(\Lambda, \mathbb{Z}_2) \cong 2^{24}$ (see [18],(2.7) and (2.10)). Hence $\tau_f\tau_e\tau_f = \varphi_\beta$ for some $\beta \in \Lambda$ and $(\tau_e\tau_f\tau_e\tau_f)^2 = (\varphi_\alpha\varphi_\beta)^2 = 1$. Therefore, $\tau_e\tau_f$ is of order $1, 2$ or $4$.

Since $xy \in 3A$, $\xi \circ \mu(xy) = t_M t_N$ has order 3 and $\langle e, f\rangle = \frac{13}{2^{10}}$. Hence $M + N \cong DIH_6(14)$ [12]. $\square$

The following lemma can be obtained easily by direct calculation (see [19], for example ).

**Lemma 3.2.** *Let $(M, N)$ be an $EE_8$-pair such that $M + N \cong DIH_6(14)$. Then*

$$\langle e_M, \varphi_b e_N\rangle = \begin{cases} \frac{13}{2^{10}} & \text{if } \langle b, M\cap N\rangle \in 2\mathbb{Z}, \\ \frac{5}{2^{10}} & \text{otherwise.} \end{cases}$$

By Lemma 3.2, we can refine the statement of Lemma 3.1 as follows.

**Proposition 3.3.** *Let $x, y, z \in \mathbb{M}$ be such that $x, y \in 2A, xy \in 3A$ and $z \in 2B \cap C_{\mathbb{M}}(x, y)$. Then, the triple $(x, y, z)$ is conjugate to $(\tau_{e_M}, \tau_{e_N}, z)$ in $\mathbb{M}$, where $(M, N)$ is an $EE_8$-pair in $\Lambda$ such that $M + N \cong DIH_6(14)$.*

**Proof.** Since $(det(M), det(\Lambda)) = 1$ and $M$ is a direct summand of $\Lambda$, there is $\beta \in \Lambda$ such that $P_M(\beta) \in a + 2M$, where $P_M$ is the natural projection from $\Lambda$ to $M^*$. Then $\varphi_\beta(\varphi_a(e_M)) = e_M$. Hence, we may assume $x = e_M$ and $y = \varphi_b(e_N)$. Since $\langle x, y\rangle = \langle e_M, \varphi_b(e_N) = \frac{13}{2^{10}}$, we have $\langle b, M \cap N\rangle \in 2\mathbb{Z}$ by Lemma (3.2) and $P_{(M\cap N)}(b) \in 2(M \cap N)^*$.

Recall that $N^* = \frac{1}{2}N$. Thus, $b = \frac{1}{2}\eta$ for some $\eta \in N$. We may assume without loss that $\eta \in ann_N(M \cap N) \mod 2N$. Let $h = t_N t_M$. Then $h$ is of order 3 and $h(N) = M$. Define $\alpha = -h(\eta) \in M$. Then $\alpha = \frac{1}{2}(\eta + (h^2 - h)\eta)$ and we have $P_N(\alpha) = b$. Thus,

$$\varphi_\beta(y) = e_N \quad \text{and} \quad \varphi_\beta(x) = \varphi_\beta(e_M) = e_M$$

as desired. □

The next theorem is important to our study, which translates a problem about $V^\natural$ to the study of the Leech lattice.

**Proposition 3.4.** *Let $(M, N)$ be an $EE_8$-pair in $\Lambda$ such that $M + N \cong DIH_6(14)$. Then*

$$\xi \circ \mu : C_{Aut(V^\natural)}(\tau_{e_M}, \tau_{e_N}, z) \to C_{O(\Lambda)}(t_M, t_N)/\langle \pm 1\rangle$$

*is surjective.*

**Proof.** Let $s \in C_{O(\Lambda)}(t_M, t_N)/\langle \pm 1\rangle$. Then by Lemma (2.11), there is $g \in Aut(V^\natural)$ such that $g$ stabilizes both $V_M^+$ and $V_N^+$ and $\xi \circ \mu(g) = s$.

By the classification of cvcc$\frac{1}{2}$ in $V_M^+$ [8, 18], $g(e_M) = \varphi_a(e_M)$ for some $a \in M^*$. By the same argument as in Proposition 3.3, we may also assume $g(e_M) = e_M$.

Since $g$ also stabilizes $V_N^+$, we have $g(e_N) = \varphi_b(e_N)$ for some $b \in N^*$. Moreover,

$$\langle e_M, \varphi_y e_N\rangle = \langle g(e_M), g(e_N)\rangle = \langle e_M, e_N\rangle = \frac{13}{2^{10}}.$$

Thus, $\langle b, M \cap N\rangle \in 2\mathbb{Z}$ by Lemma (3.2). Then by the same argument as in Proposition 3.3, there is an $\alpha \in M$ such that $P_N(\alpha) = b$. Hence

$$\varphi_\alpha(g(e_N)) = e_N \quad \text{and} \quad \varphi_\alpha(g(e_M)) = \varphi_\alpha(e_M) = e_M.$$

Therefore, $g' = \tau_{\omega^+(\alpha)} g \in C_{Aut(V^\natural)}(\tau_{e_M}, \tau_{e_N}, z)$ and $\xi \circ \mu(g') = s$. □

**Corollary 3.5.** *Let $(M, N)$ be an $EE_8$-pair in $\Lambda$ such that $M + N \cong DIH_6(14)$. The centralizer $C_{Aut(V^\natural)}(\tau_{e_M}, \tau_{e_N}, z)$ contains a subquotient isomorphic to the common stabilizer of $M$ and $N$ in $O(\Lambda)/\{\pm 1\}$.*

*Proof.* Since $C_{O(\Lambda)}(t_M, t_N)$ is the common stabilizer of $M$ and $N$ in $O(\Lambda)$, we have the conclusion by (3.4). □

## 3.1 $DIH_6(14)$

Next we shall compute the group $C_{O(\Lambda)}(t_M, t_N)$. First we recall some notations and facts about $DIH_6(14)$ from [12].

**Notation 3.6.** *1. Let $M$ and $N$ be $EE_8$-sublattices of the Leech lattice $\Lambda$ such that $Q := M + N$ is isometric to $DIH_6(14)$ as obtained in [12]. We have $F := M \cap N \cong AA_2$.*

*Let $t_M$ and $t_N$ be the SSD involutions associated to $M$ and $N$, respectively. Then the subgroup $D := \langle t_M, t_N\rangle$ generated by $t_M$ and $t_N$ is a dihedral group $Dih_6$.*

*2. Let $J := ann_Q(F)$. Then $J \cong K_{12}$ is isometric to the Coxeter-Todd lattice and $Q$ contains a sublattice isometric to $F \perp J$.*

*3. $\mathcal{D}(Q) = 2^2 \times 3^5$.*

*4. Set $g := t_M t_N$. Then $g$ has order $3$ and it acts on $\Lambda$ with trace $6$. Let $K := Fix_\Lambda(g)$ be the fixed point sublattice of $g$ in $\Lambda$. Then $ann_\Lambda(K) = J$ (see (2)). Moreover, $K \cong J \cong K_{12}$.*

*5. Let $g_1$ be an isometry of order 3 in $O(\Lambda)$ such that $g_1$ acts fixed point free on $K$ but acts trivially on $J$. In this case, $g$ and $g_1$ generate an elementary abelian group of shape $3^2$ and $gg_1$ has trace $-12$ on $\Lambda$.*

Next we recall some basic properties of the Coxeter-Todd lattice $K_{12}$. It is well-known (cf. [3, 4]) that $K_{12}$ can also be viewed as a rank 6 complex lattice over the ring of Eisenstein integers $\mathbb{Z}[\omega]$ as follows:

**Notation 3.7.** *Let $\omega := (-1+\sqrt{-3})/2$ be a primitive cubic root of unity and let $\mathcal{E} := \mathbb{Z}[\omega]$ be the ring of Eisenstein integers. Then $\mathcal{E}/2\mathcal{E} \cong \mathbb{F}_4$. Let $\sigma : \mathcal{E} \to \mathcal{E}/2\mathcal{E}$ be the natural quotient map and $\mathcal{H}$ the hexacode over $\mathcal{F}_4$. Then the Coxeter-Todd lattice can be defined as the sublattice*

$$K_{12} = \{(x_1, \ldots, x_6) \in \mathcal{E}^6 \mid (\sigma(x_1), \ldots, \sigma(x_6)) \in \mathcal{H}\}.$$

*The norm of a vector $v$ in $K_{12}$ is defined by $\langle v, v\rangle$, where $\langle\ ,\ \rangle$ is the standard Hermitian inner product on $\mathbb{C}^6$.*

By direct calculation, it is easy to show that $K_{12}$ has 756 vectors of norm 4, 4032 vectors of norm 6 and 20412 vectors of norm 8 [4].

**Notation 3.8.** *We denote the complex isometry group of $K_{12}$ by $G$. In other word, $G$ is the set of all complex linear automorphisms of $\mathbb{C}^6$ that preserve the norm and stabilize $K_{12}$.*

**Remark 3.9.** *Let $\rho$ be a root of unity in $\mathbb{Z}[\omega]$, i.e., $\rho = \pm 1, \pm\omega$ or $\pm\omega^2$, and let $\lambda_\rho$ be the linear map defined by $v \to \rho \cdot v$. Then $\lambda_\rho$ defines a complex isometry on $K_{12}$ and clearly, it is contained in the center of $G$.*

The following result can be found in [4].

**Theorem 3.10.** *Let $H$ be the subgroup generated by complex refections defined by the minimal (norm 4) vectors of $K_{12}$. Note that there are $126 (= 756/6)$ such reflections. Then*

*1. $H$ acts transitively on the sets of vectors of norms 4, 6 and 8, respectively.*

*2. $H = G$.*

*3. The order of $G$ is $2^9 \cdot 3^7 \cdot 5 \cdot 7$.*

*4. The center of $G$ is given by $\{\lambda_\rho \mid \rho = \pm 1, \pm\omega, \pm\omega^2\}$ and has order 6.*

**Remark 3.11.** *1. For each of the 756 minimal vectors $v \in K_{12}$, the sublattice $\mathbb{Z}[\omega]v \cong 2\mathbb{Z}[\omega]$ has exactly 6 minimal vectors. Moreover, $\mathbb{Z}[\omega]v \cong 2\mathbb{Z}[\omega]$ is isometric to $AA_2$ as an integral lattice. Note also that $\mathbb{Z}[\omega]v \cong 2\mathbb{Z}[\omega] \cong AA_2$*

*is invariant under the action of $\lambda_\omega$. Therefore, we have exactly 126 $\lambda_\omega$-invariant $AA_2$ sublattices in $K_{12}$ and a complex reflection on a minimal vector corresponds to a RSSD-involution associated to a $\lambda_\omega$-invariant $AA_2$-sublattice of $K_{12}$.*

*2. Let $\nu$ be the anti-automorphism defined by coordinatewise complex conjugation and let $\phi$ be the linear transformation on $\mathbb{C}^6$ defined by the matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\omega} \end{pmatrix}.$$

*Then the anti-automorphism $\nu \circ \phi$ defines an isometry of the real lattice $K_{12}$. Note that $\nu \circ \phi$ preserves the hexacode $\mathcal{H}$ and is an involution (see Proposition 4.5 of [9]). Adjoining this anti-automorphism to $G$ will give the real isometry group of $K_{12}$ [3, Section 4.9].*

The next theorem follows from Theorem 3.10 and Remark 3.11.

**Lemma 3.12.** *Let $g$ be an order 3 element in $O_3(O(K_{12}))$ and let $H$ be the subgroup generated by RSSD involutions associated to $g$-invariant $AA_2$ sublattices of $K_{12}$. Then $H$ is an index 2 subgroup of $O(K_{12})$ and its center has order 6. Moreover, $H$ acts transitively on the sets of vectors of norms 4, 6 and 8, respectively.*

We also note that the discriminant group $\mathcal{D}(K_{12}) = K_{12}^*/K_{12} \cong 3^6$. It forms a non-singular quadratic space of minus type with respect to the standard bilinear $\frac{1}{3}\mathbb{Z}/\mathbb{Z}(\cong \mathbb{F}_3)$ form [3, 4]. The isometry group $O(K_{12})$ acts on $\mathcal{D}(K_{12})$ as the orthogonal group $O^-(6,3) = \Omega^-(6,3).2^2 = 2.P\Omega^-(6,3).2^2$ and the kernel of the action is a subgroup of order 3. Since $P\Omega^-(6,3) \cong PSU(4,3)$, we have the following result.

**Lemma 3.13** (Section 4.9 of [3] and [4])**.** *Let $K_{12}$ be the Coxeter-Todd lattice of rank 12. Then*

*1. $O(K_{12})$ has the order $2^{10}.3^7.5.7$ and the shape $(6.PSU(4,3).2).2$. 2. The complex isometry group $G$ is an index 2 subgroup of $O(K_{12})$ and has the shape $6.PSU(4,3).2$.*

**Lemma 3.14.** *Let $J \cong K_{12}$ and $g$ be defined as in Notation 3.6. Let $H$ be the subgroup of $O(J)$ generated by RSSD involutions associated to $g$-invariant $AA_2$ sublattices in $J$. Then $H$ acts transitively on the set of all $g$-invariant $AA_2$ sublattices in $J$.*

*Proof.* Let $A$ and $B$ be two $g$-invariant $AA_2$ sublattices. Let $\alpha$ be a norm 4 vector of $A$. Then there exists $h \in H$ such that $h\alpha \in B$ by Lemma (3.12). Then $hA = span_{\mathbb{Z}}\{h\alpha, hg\alpha\} = B$ since $B$ is $g$-invariant and $g \in Z(H)$. $\square$

**Lemma 3.15.** *The centralizer $C_{O(\Lambda)}(g)$ is transitive on the set of all norm 4 vectors in $J$.*

*Proof.* We shall use the notion of hexacode balance to denote the codewords of the Golay code and the vectors in the Leech lattice [9, 3]. Namely, we shall arrange the index set $\Omega = \{1, 2, \ldots, 24\}$ into a $4 \times 6$ array such that the six columns form a sextet.

Since there is a unique conjugacy class of order 3 element with trace 6 on $\Lambda$ [2], we may assume

$$g = \begin{array}{|cccccc|} \hline & & & & & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \hline \end{array}$$

and

$$J = \left\{ \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ \hline \end{array} \in \Lambda \;\middle|\; x_i + y_i + z_i = 0 \text{ for all } i = 1, \ldots, 6 \right\}.$$

There are two types of norm 4 elements in $J$:

**Type I**

$$\pm\frac{1}{\sqrt{8}} \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 4 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array},$$

where the $\pm 4$'s are supported on a tetrad and on the second, third or fourth rows. There are 36 vectors of this type;

**Type II**

$$\pm\frac{1}{\sqrt{8}} \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 2 & 2 & 2 & 2 & 0 & 0 \\ -2 & -2 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}.$$

Each weight 4 codeword of the Hexacode $\mathcal{H}$ will give $2^4$ vectors of this kind and there are in total $720 (= 2^4 \times 45)$ norm 4 vectors of this type.

Recall that the subgroup of $M_{24}$ that stabilizes the standard sextet and fixes the first row is given by automorphism group of the Hexacode $Aut(\mathcal{H}) \cong 3.S_6$ [3, 9]. Let $U$ be the subgroup generated by all $\varepsilon_{\mathcal{O}}$, where $\mathcal{O}$ is a union of any two tetrads of the standard sextet. Then $U \cong 2^5$ and we obtain a subgroup $U.Aut(\mathcal{H}) \cong 2^5.(3.S_6)$, which commutes with $g$ and is transitive on each of these two types of norm 4 vectors.

Now it remains to show there is an element in $C_{O(\Lambda)}(g)$ which mixes these two types of vectors.

Let $\mathcal{T} = \{T_1, \ldots, T_6\}$ be a sextet given as below:

$$\begin{array}{|c|c|c|c|c|c|} \hline * & \circ & \bullet & \bullet & \bullet & \bullet \\ \hline \circ & * & \times & \times & \times & \times \\ \hline \circ & * & \diamond & \diamond & \diamond & \diamond \\ \hline \circ & * & & & & \\ \hline \end{array}.$$

We take $T_1$ to be the tetrad marked by $*$.

Let $\xi_{\mathcal{T}}$ be the linear map defined by

$$\begin{aligned} v_i &\to v_i - \frac{1}{2} v_{T_1} \quad \text{ if } i \in T_1, \\ v_i &\to \frac{1}{2} v_T - v_i \quad \text{ if } i \in T, T \in \mathcal{T}, T \neq T_1, \end{aligned}$$

where $\{\pm v_i \mid i \in \Omega = \{1, \ldots, 24\}\}$ is a standard frame of norm 8 vectors in $\Lambda$ and $v_S = \sum_{i \in S} v_i$ for any $S \subset \Omega$. Then $\xi_{\mathcal{T}}$ is an isometry of $\Lambda$ (cf. [3, p. 288] and [9, p. 97]).

It is easy to see that $\xi_{\mathcal{T}}$ commutes with $g$ and that

$$\xi_{\mathcal{T}} \left( \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & -4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 2 & 2 & -2 \\ \hline 0 & 0 & -2 & -2 & -2 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}.$$

Hence $C_{O(\Lambda)}(g)$ is transitive on the set of all norm 4 vectors in $J$. $\square$

**Lemma 3.16** (cf. [3, 9])**.** *The centralizer $C_{O(\Lambda)}(g)$ is transitive on the set $\mathcal{A} = \{A \subset J \mid A \cong AA_2$ and is $g$-invariant$\}$.*

*Proof.* Let $A, B \in \mathcal{A}$ and let $\alpha \in A$. Then by Lemma (3.15), there exists $h \in C_{O(\Lambda)}(g)$ such that $h\alpha \in B$. Since $h$ commutes with $g$ and $B$ is $g$-invariant, we have

$$hA = span_{\mathbb{Z}}\{h\alpha, hg\alpha\} = span_{\mathbb{Z}}\{h\alpha, gh\alpha\} = B$$

as desired. $\square$

**Lemma 3.17.** *Let $A$ be a $g$-invariant $AA_2$-sublattice in $J$. There exists an $EE_8$-sublattice $E$ of $\Lambda$ such that $E \cap J = A$. In this case, $E \cap K \cong EE_6$ (3.6).*

*Proof.* Clearly, there exists an $EE_8$-sublattice $X$ such that $X \cap J \cong AA_2$ (cf. Case 1: $DIH_6(14)$ of Appendix A). By Lemma (3.16), $C_{O(\Lambda)}(g)$ is transitive on $\mathcal{A} := \{A \subset J \mid A \cong AA_2$ and is $g$-invariant$\}$. Thus, there is an $h \in C_{O(\Lambda)}(g)$ such that $h(X \cap J) = A$. Now take $E = hX$. Then $E \cap J = hX \cap J = h(X \cap J) = A$ since $hJ = J$. $\square$

**Remark 3.18.** *Let $E^1$ and $E^2$ be $EE_8$ sublattices of $\Lambda$ such that $E^1 \cap J = E^2 \cap J \cong A$. Then $E^1 + E^2 \cong DIH_6(14)$ (cf. [12]). Moreover, $t_{E^1}t_{E^2}$ acts trivially on $J$ and hence $t_{E^1}t_{E^2} \in \langle g_1 \rangle$. Therefore, $E^2 = g_1^i E^1$ for some $i = 1, 2$.*

**Theorem 3.19.** *Let $M, N$ be $EE_8$-sublattices of $\Lambda$ such that $M + N \cong DIH_6(14)$. Then the pair $(M, N)$ is unique, up to the action of $O(\Lambda)$.*

*Proof.* Let $(M, N)$ be such a pair. Then $A = M \cap N \cong AA_2$ and $g = t_M t_N$ has order 3 and trace 6 on $\Lambda$. Hence $K = Fix_\Lambda(g) \cong K_{12}$ and $J = ann_\Lambda(K) \cong K_{12}$. Since such a $g$ is unique up to conjugacy, $K$ and $J$ are uniquely determined, up to the action $O(\Lambda)$.

Now note that $A \subset K$ since $A$ is the common $(-1)$-eigenlattice of $t_M$ and $t_N$. By Lemma (3.16), $(A, J)$ is unique, up to the action of $O(\Lambda)$. Now by Lemma (3.17) and Remark (3.18), the pair $(M, N)$ is unique up to the action of $O(\Lambda)$. $\square$

**Lemma 3.20** ([9])**.** *Let $g \in O(\Lambda)$ be an element of order 3 and trace 6. Then $C_{O(\Lambda)}(g)$ has the shape $(2 \times 3^2).PSU(4,3).2$.*

*Proof.* First we note that $C_{O(\Lambda)}(g)$ stabilizes both $K = Fix_\Lambda(g)$ and $J = ann_\Lambda(K)$ and hence $C_{O(\Lambda)}(g)$ acts on $J$ and induces a group homomorphism $\varphi : C_{O(\Lambda)}(g) \to C_{O(J)}(g) \cong H$.

Now let $A \in \mathcal{A}$. Then there is an $EE_8$-sublattice $E$ of $\Lambda$ such that $E \cap J = A$. In this case, the SSD involution $t_E$ acts as $t_A$ on $J$. Since the RSSD involutions $t_A, A \in \mathcal{A}$, generate $H$, we have $Im\varphi = H$.

**Claim:** $ker\varphi = \langle g_1 \rangle$.

Proof. Let $\sigma \in O(\Lambda)$ such that $\phi(\sigma) = id_J$, i.e., $\sigma$ fixes $J$ pointwise. Thus $\sigma$ acts trivially on $J^*/J$. Moreover, $\sigma$ acts on $K$ and must act trivially on $K^*/K$ since $\sigma$ preserves the gluing map from $K \perp J$ to $\Lambda$. By the

discussion before Lemma (3.13), $\sigma \in \langle g_1 \rangle$. Therefore, $C_{O(\Lambda)}(g)/\langle g_1 \rangle \cong H \cong 6.PSU(4,3).2$ and $C_{O(\Lambda)}(g)$ has the shape $(2 \times 3^2).PSU(4,3).2$. $\square$

**Remark 3.21.** *We shall note that for any $EE_8$-sublattice $E$ of $\Lambda$ such that $E \cap J \in \mathcal{A}$, the SSD involution $t_E$ inverts $g_1$ and so $t_E$ and $g_1$ generate a subgroup isomorphic to $Sym_3$.*

**Lemma 3.22.** *Let $g$ and $J$ be defined as in Notation (3.6). Then the centralizer $C_{O(\Lambda)}(g)$ is transitive on the set*

$$\mathcal{B} = \{A \perp B \subset J \mid A, B \text{ are } g\text{-invariant } AA_2 \text{ sublattices of } J\}.$$

*Proof.* Let $H$ be the subgroup generated by RSSD involutions associated to $g$-invariant $AA_2$ sublattices as defined in Lemma (3.12). Then $H$ is an index 2 subgroup of $O(J)$. By Lemma (3.20), the image of $C_{O(\Lambda)}(g)$ in $O(J)$ is $H \cong 6.PSU(4,3).2$.

Recall that $J/(g-1)J \cong J^*/J \cong 3^6$ is a non-singular quadratic space of $(-)$-type. The group $O(J)$ acts on $J/(g-1)J$ as the full orthogonal group $O^-(6,3) \cong 2.PSU(4,3).2^2$ while $C_{O(\Lambda)}(g)$ acts on $J/(g-1)J$ as $2.PSU(4,3).2$.

In $J/(g-1)J$, the image of a $g$-invariant $AA_2$ sublattice is a non-singular 1-space and the image of $g$-invariant $AA_2 \perp AA_2$ is a definite 2-space. By Witt Theorem, $O^-(6,3)$ is transitive on definite 2-spaces. Therefore, $O(J)$ is transitive on $\mathcal{B}$.

Recall from Notation (3.7) that

$$(2\mathbb{Z}[\omega])^6 \subset J \subset \mathbb{Z}[\omega]^6.$$

Let $A \perp B$ be the sum of the first and fourth copies of $2\mathbb{Z}[\omega] \cong AA_2$. Then the anti-automorphism $\nu \circ \phi$ defined in Remark (3.11) gives an isometry of the real lattice $K_{12}$ and by definition, it stabilizes $A \perp B$. Hence the index $[Stab_{O(J)}(A \perp B) : Stab_H(A \perp B)] = 2$ and $C_{O(\Lambda)}(g)$ is transitive on $\mathcal{B}$. $\square$

**Theorem 3.23.** *Let $M$ and $N$ be defined as in Notation (3.6) and let $D$ be the dihedral group generated by $t_M$ and $t_N$. Then $C_{O(\Lambda)}(D) \cong (3 \times 2 \times PSU(4,2)).2$.*

*Proof.* Set $G = C_{O(\Lambda)}(D)$. Since $g \in D$, $G$ stabilizes $K = Fix_\Lambda(g)$ and $J = ann_\Lambda(K)$. In addition, $G$ centralizes $t_M$ and hence it stabilizes the $(-1)$-eigenlattice of $t_M$, which is $M$. Therefore, $G$ acts on $F = M \cap K \cong AA_2$ and $M \cap J \cong EE_6$.

Let $\alpha \in M \cap J \cong EE_6$ be a norm 4 vector and $A = span_{\mathbb{Z}}\{\alpha, g\alpha\}$. Let $E$ be an $EE_8$ sublattice such that $E \cap J = A$. Then $t_E$ acts as $t_{\langle\alpha\rangle}$ on $M \cap J$, which is a reflection at $\alpha$. Since $M \cap E \neq 0$ and $t_E$ commutes with $t_M$, we have $M \cap E \cong AA_1^2$ or $DD_4$. Thus, $E \cap F = E \cap M \cap K \cong AA_1$ and $t_E$ acts as a reflection at a root, also. Moreover, the $-1$ map clearly acts on both $M \cap J$ and $F$. Thus, $G$ acts on $M \cap J$ as the full isometry group $O(EE_6) = 2 \times Weyl(E_6) \cong 2 \times PSU(4,2).2$ and acts as $O(AA_2) \cong 2.S_3$ on $F = M \cap K$. Hence $G = C_{O(\Lambda)}(D)$ has the shape $(3 \times 2 \times PSU(4,2)).2$. $\square$

**Corollary 3.24.** *Let $D$ be the dihedral group generated by $t_M$ and $t_N$. Then $C_{O(\Lambda)}(D)/O_2(C_{O(\Lambda)}(D)) \cong (3 \times PSU(4,2)).2$, which is isomorphic to "half" of the Weyl group of type $A_2 + E_6$.*

The above results may be lifted to a statement about the Monster as follows.

**Theorem 3.25.** *Let $(x, y, z)$ be a triple of elements in the Monster such that $x, y \in 2A$, $xy \in 3A$, $z \in 2B \cap C_{\mathbb{M}}(x, y)$. Then $(x, y, z)$ is unique up to the conjugation of $\mathbb{M}$. Moreover, $C_{\mathbb{M}}(x, y, z)$ has a homomorphism onto $C_{O(\Lambda)}(D)/O_2(C_{O(\Lambda)}(D))$, which is isomorphic to "half" of the Weyl group of type $A_2 + E_6$.*

**Proof.** That the triple is unique up to conjugation follows from Proposition 3.3 and 3.19. The last statement follows from Corollary 3.24, Proposition (3.4) and Corollary (3.5).$\square$

## 3.2 Isometry groups of $Q \cong DIH_6(14)$ and $ann_\Lambda(Q)$

Let $Q \cong DIH_6(14)$ and $R = ann_\Lambda(Q)$. We shall determine the isometry group of $Q$ and $R$ in this subsection.

**Notation 3.26.** *Let $M$ and $N$ be $EE_8$ sublattices of $\Lambda$ such that $Q = M + N \cong DIH_6(14)$ and let $R = ann_\Lambda(Q)$.*

*Let $g = t_M t_N \in O(\Lambda)$. Set $K := Fix_\Lambda(g)$, $J := ann_\Lambda(Fix_\Lambda(g))$ and $F := M \cap N$. Then $K \cong J \cong K_{12}$, $F \cong AA_2$ and $F$ is orthogonal to $J$. In addition, we denote the order 3 fixed point free isometry in $O_3(O(K))$ by $g_1$.*

We shall use the embedding of $Q$ and $R$ in $\Lambda$ as discussed in Appendix A. Then $R$ can be obtained by gluing $AA_2^5$ with the glue code

$$\mathcal{C} = span_{\mathbb{F}_4}\{(1,1,1,1,0), (1,0,\omega,\omega^2,1)\}.$$

Note that $\mathcal{C}$ has 15 codewords of weight 4 and $|R(4)| = 15 \times 2^6 + 5 \cdot 6 = 270$. Moreover, $ann_R(A) \cong A_2 \otimes D_4$ for an $AA_2$-sublattice $A$ in $AA_2^5$.

**Lemma 3.27.** *$O(A_2 \otimes D_4) \cong O(A_2) \circ O(D_4)$, which has order $2^8 \cdot 3^3$.*

*Proof.* It is clear that $O(A_2 \otimes D_4)$ has a subgroup isomorphic to $O(A_2) \circ O(D_4)$. Since the minimal vectors of $A_2 \otimes D_4$ have the form $\alpha \otimes \beta$, where $\alpha$ and $\beta$ are roots of $A_2$ and $D_4$, respectively [12], we have $O(A_2 \otimes D_4) \cong O(A_2) \circ O(D_4)$ □

**Lemma 3.28.** *The order of $O(R)$ is $2^8 \cdot 3^5 \cdot 5$.*

*Proof.* Since $g_1$ is fixed point free on $R$, $R$ has $45 (= 270/6)$ distinct $g_1$-invariant $AA_2$ sublattices.

Now let $A$ be an $AA_2$-sublattice in $AA_2^5$. Then the stabilizer $Stab_{O(R)}(A)$ of $A$ acts on the sublattice $R' := ann_R(A) \cong A_2 \otimes D_4$. Let $\varphi : \mathcal{D}(A) \to \mathcal{D}(R')$ be the gluing map. Then

$$Stab_{O(R)}(A) = \{(h, h') \in O(A) \times O(R') \mid \varphi \circ h = h' \circ \varphi\}.$$

Let $p_{R'} : Stab_{O(R)}(A) \to O(R') \cong O(A_2) \circ O(D_4)$ be the restriction map. Then $ker(p_{R'}) = \langle t_A \rangle \cong \mathbb{Z}_2$ and $Im(p_{R'})$ is an index 2 subgroup of $O(R')$. Note that the field automorphism $\omega \to \omega^2$ defines an isometry on $R'$ but it does not lift to $O(R)$.

Therefore, $|Stab_{O(R)}(A)| = (|O(R')| \cdot 2)/2 = 2^8 \cdot 3^3$ and hence $|O(R)| = |Stab_{O(R)}(A)| \cdot (45) = 2^8 \cdot 3^5 \cdot 5$. □

**Proposition 3.29.** *The isometry group $O(R)$ has the shape $6.PSU(4,2).2$.*

*Proof.* First, we note that $R = ann_{K_{12}}(A)$ for an $AA_2$ sublattice $A$ of $K_{12}$. Hence, $Stab_{O(K_{12})}(A) = C_{O(K_{12})}(t_A)$ acts on $R$ with the kernel $\langle t_A \rangle$.

Since $O(K_{12}) \cong 6.PSU(4,3).2^2$, by [2, Page 52,53],

$$C_{O(K_{12})}(t_A) \cong 6.PSU(4,2).2 \times 2.$$

Hence $C_{O(K_{12})}(t_A)/\langle t_A \rangle \cong 6.PSU(4,2).2$ acts faithfully on $R$.

Since $|6.PSU(4,2).2| = 2^8 \cdot 3^5 \cdot 5$, $O(R)$ has the shape $6.PSU(4,2).2$ by Lemma (3.28). □

**Theorem 3.30.** *Let $Q \cong DIH_6(14)$ and $J$ be defined as in (3.26). Let $\varphi : O(Q) \to O(J)$ be the restriction map. Then $Im(\varphi) \cong O(A_2) \circ O(E_6)$ and $Ker(\varphi) \cong Sym_3$. Therefore, $O(Q)$ has the shape $Sym_3.(O(A_2) \circ O(E_6))$.*

*Proof.* Let $A$ and $J$ be defined as in (3.26). Then $O(Q)$ stabilizes both $A$ and $J$. Hence, by Lemma (A.7), $O(Q)$ stabilizes the set $\{M \cap J, gM \cap J, g^2M \cap J\}$ and the sublattice $M \cap J + gM \cap J \cong A_2 \otimes E_6$. Recall that $M \cap J \cong EE_6$.

By Theorem (3.23), $O(Q)$ acts as the full isometry group $O(E_6)$ on the sublattices $M \cap J$, $gM \cap J$ and $g^2M \cap J$, respectively. On the other hand, $\langle t_M, t_{gM}\rangle \cong Dih_6$ acts as permutations on the set $\{M \cap J, gM \cap J, g^2M \cap J\}$. Thus, we have $Im(\varphi) \cong O(A_2 \otimes E_6) = O(A_2) \circ O(E_6)$.

Clearly, $Ker(\varphi)$ can be viewed as a subgroup of $O(A) \cong \langle t_A\rangle.Weyl(A_2)$. Since $[Q : A \perp J] = 3$, $A$ is not an RSSD in $Q$ and thus $t_A$ is not an isometry of $Q$. Hence we have $Ker(\varphi) < Weyl(A_2)$.

Let $p_J : Q \to \mathcal{D}(J)$ and $p_A : Q \to \mathcal{D}(A) \cong 2^2 \cdot 3$ be the natural maps. Let $\xi : p_A(Q) \to p_J(Q)$ be the gluing map from $A \perp J$ to $Q$. Since $O(Q)$ stabilizes $A \perp J$, we have

$$O(Q) = \{y \in O(A) \times O(J) \mid y \text{ preserves the gluing map, } i.e., y\xi = \xi y\}.$$

Thus $y \in Ker(\varphi)$ if and only if $y \in O(A) \times O(J)$ acts trivially on $J$ and $y\xi = \xi y$. Therefore, we have $Ker(\varphi) = \{y \in O(A) \mid \xi = \xi y\}$.

By the discussion in Case 1 of Appendix A, we know that $p_A(Q) = 2A^*/A$ is a subgroup of order 3. Since $Weyl(A_2)$ fixes all cosets in $2AA_2^*/AA_2$, we have $Ker(\varphi) \cong Weyl(A_2) \cong Sym_3$. $\square$

**Remark 3.31.** *We note that $\mathbb{Z}\alpha$ is an RSSD sublattice in $Q$ for any norm 4 vector $\alpha \in A$. The reason is as follows:*

*Let $\alpha \in A$ with $(\alpha, \alpha) = 4$. We want to show that $2Q \leq ann_Q(\alpha) + \mathbb{Z}\alpha$. That is equivalent to $(\alpha, Q) \leq 2\mathbb{Z}$. First we notice that the index $|Q : J+A| = 3$ since $\mathcal{D}(J) \cong 3^6$, $\mathcal{D}(A) \cong 2^2 \times 3$ and $det(Q) = 2^2 3^5$. Moreover, we have $(\alpha, J + A) \leq 2\mathbb{Z}$ since $A \cong AA_2$ is doubly even. Also, there is an integer $r$ so that $(\alpha, Q) = r\mathbb{Z}$. Since $3Q \leq J + A$, $3r\mathbb{Z} \leq 2\mathbb{Z}$, whence $r$ is even. We conclude that $(\alpha, Q) \leq 2\mathbb{Z}$.*

# 4 6A-triples

In this section, we consider a triple $(x, y, z)$ in $\mathbb{M}$ such that $x, y \in 2A$, $xy \in 6A$ and $z \in 2B \cap C_{\mathbb{M}}(x, y)$. We shall study the orbits of such triples under the action of $\mathbb{M}$.

**Notation 4.1.** *Let $\mathcal{SA} = \{(x, y, z) \mid x, y \in 2A, xy \in 6A, z \in 2B \cap C_{\mathbb{M}}(x, y)\}$. Note that the Monster $\mathbb{M}$ acts on $\mathcal{SA}$ by conjugation.*

Take $(x, y, z) \in \mathcal{SA}$. Then $z \in 2B$ and we may again assume $z$ acts as 1 on $V_\Lambda^+$ and as $-1$ on $V_\Lambda^{T,+}$ by conjugation. Moreover, $x = \tau_e$ and $y = \tau_f$ for some cvcc$\frac{1}{2}$ $e$ and $f$ in $V_\Lambda^+$, by the Miyamoto bijection [15, 21] . By our assumption, $xy \in 6A$ and thus $(xy)^3 \in 2A$. There are two cases:

1. $(xy)^3 \in O_2(C_{\mathbb{M}}(z)) \cong 2^{1+24}$;
2. $(xy)^3 \notin O_2(C_{\mathbb{M}}(z))$.

**Case 1** $(6A.1)$**:** $(xy)^3 \in O_2(C_{\mathbb{M}}(z)) \cong 2^{1+24}$.

In this case, $\xi \circ \mu(xy)$ has order 3 in $O(\Lambda)/\{\pm 1\}$. Thus, by the same arguments as in (3.1) and (3.3), we may assume $e = e_M$ and $f = \varphi_b(e_N)$ for some $EE_8$-pair $(M, N)$ and $b \in N^*$ such that $M + N \cong DIH_6(14)$. Since $xy \in 6A$, we have $\langle e, f\rangle = \frac{5}{2^{10}}$ and hence $\langle b, M \cap N\rangle \notin 2\mathbb{Z}$ by (3.2). Then $b = \frac{1}{2}\alpha \mod 2N$ for some $\alpha \in M \cap N(4)$.

**Theorem 4.2.** *Let $(x, y, z) \in \mathcal{SA}$. Suppose $(xy)^3 \in O_2(C_{\mathbb{M}}(z))$. Then $(x, y, z)$ is conjugate to $(\tau_{e_M}, \tau_{\varphi_{\alpha/2}(e_N)}, z)$ for some $EE_8$-pair $(M, N)$ such that $M + N = DIH_6(14)$ and $\alpha \in M \cap N(4)$.*

**Proposition 4.3.** *Let $(M, N)$ be an $EE_8$-pair in $\Lambda$ such that $M + N \cong DIH_6(14)$. Let $\alpha \in N(4)$ such that $\langle \frac{1}{2}\alpha, M \cap N\rangle \notin 2\mathbb{Z}$. Then*

$$\xi \circ \mu : C_{Aut(V^\natural)}(\tau_{e_M}, \tau_{\varphi_{\frac{\alpha}{2}(e_N)}}, z) \to C_{Stab_{O(\Lambda)}(\mathbb{Z}\alpha)}(t_M, t_N)/\langle \pm 1\rangle$$

*is surjective.*

*Proof.* By Lemma (3.2), $\langle e_M, \varphi_{\frac{\alpha}{2}}(e_N)\rangle = \frac{5}{2^{10}}$ and $\tau_{e_M}, \tau_{\varphi_{\frac{\alpha}{2}(e_N)}}$ generate a dihedral group of order 12 in $Aut(V^\natural)$. In this case, $(\tau_{e_M}\tau_{\varphi_{\frac{\alpha}{2}(e_N)}})^3 = \tau_{\omega^+(\alpha)}$ or $\tau_{\omega^-(\alpha)}$ [19]. Thus, the subgroup generated by $\tau_{e_M}, \tau_{\varphi_{\frac{\alpha}{2}(e_N)}}, z$ is the same as the group generated by $\tau_{e_M}, \tau_{e_N}, \tau_{\omega^+(\alpha)}$ and $z$. Recall that $\tau_{\omega^+(\alpha)}z = \tau_{\omega^-(\alpha)}$ [18]. The result now follows by Lemma (2.9) and Proposition (3.4). □

**Case 2** $(6A.2)$**:** $(xy)^3 \notin O_2(C_{\mathbb{M}}(z))$.

Then $\xi \circ \mu(xy)$ has order 6 in $O(\Lambda)/\{\pm 1\}$. By the analysis in [12], $e = \varphi_a(e_M)$ and $f = \varphi_b(e_N)$ for some $EE_8$-pair $(M, N)$ in $\Lambda$ such that $M + N \cong DIH_{12}(16)$ and $a \in M^*$, $b \in N^*$

As in Proposition 3.3, we may also assume $e = e_M$, up to conjugation. Since $M \cap N = 0$ and $DIH_{12}(16)$ is a direct summand of $\Lambda$, there is a $\beta \in \Lambda$ such that $P_N(\beta) = b \mod 2N$ and $\langle P_M(\beta), M\rangle \in 2\mathbb{Z}$ by (D.2). Then, $\varphi_\beta(f) = \varphi_\beta(\varphi_b(e_N)) = e_N$ and $\varphi_\beta(e) = \varphi_\beta(e_M) = e_M$.

Thus, up to conjugation, $e = e_M$ and $f = e_N$ for some $EE_8$-pair $(M, N)$ such that $M + N \cong DIH_{12}(16)$.

**Theorem 4.4.** *Let $(x, y, z) \in \mathcal{SA}$. Suppose $(xy)^3 \notin O_2(C_{\mathbb{M}}(z))$. Then $(x, y, z)$ is conjugate to $(\tau_{e_M}, \tau_{e_N}, z)$ for some $EE_8$-pair $(M, N)$ such that $M + N \cong DIH_{12}(16)$.*

**Proposition 4.5.** *Let $(M, N)$ be an $EE_8$-pair in $\Lambda$ such that $M + N \cong DIH_{12}(16)$. Then*

$$\xi \circ \mu : C_{Aut(V^\natural)}(\tau_{e_M}, \tau_{e_N}, z) \to C_{O(\Lambda)}(t_M, t_N)/\langle \pm 1 \rangle$$

*is surjective.*

**Proof.** Let $s \in C_{O(\Lambda)}(t_M, t_N)/\langle \pm 1 \rangle$. Then by Lemma (2.11), there is $g \in Aut(V^\natural)$ such that $g$ stabilizes both $V_M^+$ and $V_N^+$ and $\xi \circ \mu(g) = s$.

As in Proposition 3.4, we may assume $g(e_M) = e_M$ and $g(e_N) = \varphi_b(e_N)$ for some $b \in N^*$. Since $M \cap N = 0$ and $DIH_{12}(16)$ is a direct summand of $\Lambda$, there is a $\beta \in \Lambda$ such that $P_N(\beta) = b \mod 2N$ and $\langle P_M(\beta), M \rangle \in 2\mathbb{Z}$ by (D.2). Then, $\varphi_\beta(g(e_N)) = \varphi_\beta(\varphi_b(e_N)) = e_N$ and $\varphi_\beta(g(e_M)) = \varphi_\beta(e_M) = e_M$

Therefore, $g' = \tau_{\omega^+(\beta)} g \in C_{Aut(V^\natural)}(\tau_{e_M}, \tau_{e_N}, z)$ and $\xi \circ \mu(g') = s$. □

## 4.1 Case $6A.1$ and $M + N \cong DIH_6(14)$

First we consider the case $6A.1$. In this case, $M + N \cong DIH_6(14)$ and $e = e_M$ and $f = \varphi_{\frac{\alpha}{2}}(e_N)$ for some $\alpha \in (M \cap N)(4)$.

**Proposition 4.6.** *Let $M$ and $N$ be defined as in Notation (3.6) and let $\alpha \in M \cap N$ be a norm 4 vector. Then $C_{Stab_{O(\Lambda)}(\mathbb{Z}\alpha)}(t_M, t_N) \cong 2 \times PSU(4, 2).2$.*

*Proof.* First, we note that $t_M, t_N$ stabilize $\mathbb{Z}\alpha$ since $\alpha \in M \cap N$. Moreover, the group

$$C_{Stab_{O(\Lambda)}(\mathbb{Z}\alpha)}(t_M, t_N) = \{h \in C_{O(\Lambda)}(t_M, t_N) \mid h(\mathbb{Z}\alpha) = \mathbb{Z}\alpha\}.$$

By Theorem (3.23), $C_{O(\Lambda)}(t_M, t_N)$ has the shape $2.(3 \times PSU(4, 2)).2$ which acts as $2.S_3$ on $M \cap N$ and acts as the isometry group of $ann_M(M \cap N) \cong EE_6$. Thus, the subgroup that fixes $\mathbb{Z}\alpha$ has the shape $2 \times PSU(4, 2).2$. □

By Lemma (2.11) and Proposition (4.3), we have the corollary.

**Corollary 4.7.** *Let $M$, $N$ and $\alpha$ be defined as in Proposition (4.6). Let $z$ be the automorphism of $V^\natural$ such that $z|_{V_\Lambda^+} = 1$ and $z|_{V_\Lambda^{T,+}} = -1$ as defined as*

*in (2.8). Then there is a homomorphism that maps $C_{Aut(V^\natural)}(\tau_{e_M}, \tau_{\varphi_{\frac{\alpha}{2}}(e_N)}, z)$ onto*

$$C_{Stab_{O(\Lambda)}(\mathbb{Z}\alpha)}(t_M, t_N)/O_2(C_{Stab_{O(\Lambda)}(\mathbb{Z}\alpha)}(t_M, t_N)) \cong PSU(4,2).2.$$

*The kernel $K$ is a 2-group of order $2^{11}$ and we have an exact sequence*

$$1 \to \langle z \rangle \to K \to \{\varphi_\beta \mid \beta \in \Lambda \text{ and } \langle \beta, M+N \rangle \in 2\mathbb{Z}\} \to 1.$$

**Proof.** To compute the kernel, note that the natural map $\Lambda \to Hom(M+N, \mathbb{Z}_2)$ is onto since $M+N$ is a direct summand and $det(\Lambda) = 1$. □

By Theorem 4.2 and Corollary 4.7, we have our main theorem as follows.

**Theorem 4.8.** *Let $(x, y, z)$ be a triple of elements in the Monster such that $x, y$ in $2A$, $xy$ in $6A$, $z \in 2B \cap C_{\mathbb{M}}(x,y)$ and $(xy)^3 \in O_2(C_{\mathbb{M}}(z))$. Then such triples form one orbit under the conjugation action of $\mathbb{M}$ and $C_{\mathbb{M}}(x,y,z)$ has a homomorphism onto*

$$C_{Stab_{O(\Lambda)}(\mathbb{Z}\alpha)}(t_M, t_N)/O_2(C_{Stab_{O(\Lambda)}(\mathbb{Z}\alpha)}(t_M, t_N)) \cong PSU(4,2).2.$$

*The kernel $K$ is a 2-group of order $2^{11}$ and we have an exact sequence*

$$1 \to \langle z \rangle \to K \to \{\varphi_\beta \mid \beta \in \Lambda \text{ and } \langle \beta, M+N \rangle \in 2\mathbb{Z}\} \to 1.$$

## 4.2 Case $6A.2$ and $M + N \cong DIH_{12}(16)$

Next we consider the case $6A.2$. In this case, $M + N \cong DIH_{12}(16)$, $e = e_M$ and $f = e_N$.

**Notation 4.9** ([12])**.** *1. Let $M$ and $N$ be $EE_8$ sublattice of $\Lambda$ such that $Q' := M + N$ is isometric to $DIH_{12}(16)$ (Table 1 of [12]). Then $ann_M(N) \cong ann_N(M) \cong AA_2$. We also denote $R' := ann_\Lambda(Q')$.*

*2. Let $F' := ann_M(N) \perp ann_N(M)$ and $J := ann_{Q'}(F')$. Then $J \cong K_{12}$ is isometric to the Coxeter-Todd lattice and $Q'$ contains a sublattice isometric to $F' \perp J$.*

*3. $\mathcal{D}(Q') \cong \mathcal{D}(R') \cong 6^4$.*

By explicit calculation in the Leech lattice (see Appendix A), one can show that $R'$ contains a sublattice isometric to $AA_2^{\perp 4}$ and $R'$ is isometric to

$$span_{\mathbb{Z}}\{AA_2^{\perp 4}, \frac{1}{2}(\beta_1, \beta_1, \beta_1, \beta_1), \frac{1}{2}(\beta_2, \beta_2, \beta_2, \beta_2)\},$$

where $\beta_1 = \sqrt{2}\alpha_1$, $\beta_2 = \sqrt{2}\alpha_2$ and $\{\alpha_1, \alpha_2\}$ is a set of fundamental roots for $A_2$. In fact, $R' \cong A_2 \otimes D_4$ (see (A.1)).

**Notation 4.10** ([12]). *Let $t_M$ and $t_N$ be the SSD involutions associated to $M$ and $N$. Then the group $\Delta := \langle t_M, t_N\rangle \cong Dih_{12}$. Set $h := t_M t_N$, $g := h^2$ and $u = h^3$. Then $h$ has order 6, $g$ has order 3 and $u$ has order 2. The traces of $h$, $g$ and $u$ on $\Lambda$ are 2, 6 and 8, respectively. Note also that $\langle u\rangle = Z(\Delta)$ and $g = uh^{-1}$.*

**Proposition 4.11.** *Let $(M, N)$ be an $EE_8$-pair in $\Lambda$ such that $M + N \cong DIH_{12}(16)$. Then the pair $(M, N)$ is unique up to the action of $O(\Lambda)$.*

*Proof.* Let $(M, N)$ be such a pair. Then $g = (t_M t_N)^2$ has order 3 and trace 6 on $\Lambda$. Let $K := Fix_\Lambda(g) \cong K_{12}$ and $J := ann_\Lambda(K) \cong K_{12}$. Since such a $g$ is unique up to conjugacy, $K$ and $J$ are uniquely determined, up to the action $O(\Lambda)$.

Let $g'$ be the fixed point free order 3 element in $O_3(O(K))$. Then, $F_M := M \cap K$ and $F_N := N \cap K$ are both $g'$-invariant $AA_2$-sublattices in $K$ and $F_M$ is orthogonal to $F_N$. By Lemma (3.22), $F_M \perp F_N \perp J$ is unique up to the action of $O(\Lambda)$. Since $M + N$ is a direct summand of $\Lambda$, we have $ann_\Lambda(ann_\Lambda(F_M \perp F_N \perp J)) = M + N \cong DIH_{12}(16)$.

By Lemma (3.17), there exists $EE_8$ sublattices $E^1$ and $E^2$ such that $E^1 \cap K = F_M$ and $E^2 \cap K = F_N$. Then $E^1 + E^2 \cong DIH_{12}(16)$ (cf. [12]) and thus $E^1 + E^2 = M + N$. Therefore, $E^1 = g^i M$ and $E^2 = g^j N$ for some $i, j = 0, 1, 2$ (see (A.6)) and $(E^1, E^2)$ is conjugate to $(M, N)$ by the action of the dihedral group $\langle t_M, t_N\rangle$. Hence $(M, N)$ is unique up to the action of $O(\Lambda)$. $\square$

Next we recall few facts about the lattice $DIH_{12}(16)$ from [12].

**Lemma 4.12** ([12]). *Let $h$, $g$ and $u$ be defined as in Notation (4.10). Then*

*1. $N \cap g^{-1}M \cong DD_4$ and $N + g^{-1}M \cong DIH_4(12)$.*

*2. $u = t_N t_{g^{-1}M}$ is an SSD involution associated to an $EE_8$ sublattice $E$ of $N + g^{-1}M$*

**Lemma 4.13** (Proposition 6.44 of [12]). *Let $P = g^{-1}M \cap J$ and let $P^-(t_N)$ be the $(-1)$-eigenspace of $t_N$ in $P$. Then $P^-(t_N) = N \cap g^{-1}M \cong DD_4$ and $N \cap g^{-1}M < J$.*

**Lemma 4.14.** *Let $\Delta = \langle t_M, t_N \rangle$ be defined as in Notation (4.10). The centralizer $C_{O(\Lambda)}(\Delta)$ has the shape*

$$(3 \times 2 \times 2.(Alt_4 \times Alt_4).2).2,$$

*which is an index 2 subgroup of* $2.(Dih_6 \times O(D_4))$.

*Proof.* Set $G_1 := C_{O(\Lambda)}(\Delta)$.

Let $L = M + gM$. Then $L \cong DIH_6(14)$ and the group $\langle t_M, t_{gM} \rangle \cong Dih_6$ is a subgroup of $\Delta$. By the analysis of $DIH_6(14)$, the centralizer

$$C_1 := C_{O(\Lambda)}(\langle t_M, t_{gM} \rangle) \cong (3 \times 2 \times PSU(4,2)).2 \qquad (5)$$

and it stabilizes $g^iM \cap J$ for each $i = 0, 1, 2$ and acts as $O(EE_6)(\cong 2 \times PSU(4,2).2)$ on each $g^iM \cap J$, where $J = ann_\Lambda(K)$ and $K = Fix_\Lambda(g)$.

Note that $G_1$ also centralizes $t_N$. Thus $G_1$ stabilizes $N$ and $N \cap g^{-1}M \cong DD_4$. By Lemma 4.13, $N \cap g^{-1}M < J \cap g^{-1}M \cong EE_6$. Therefore, $G_1$ acts as the stabilizer of $N \cap g^{-1}M \cong DD_4$ on $g^{-1}M \cap J$, which is isomorphic to $2.O(DD_4) \cong 2.O(D_4)$. Note that $O(D_4) \cong Weyl(F_4) \cong O^+(4,3)$ and has the shape $2.(Alt_4 \times Alt_4).2^2$ (see (B.3) and Appendix B of [14]).

Since $u$ commutes with $D = \langle t_M, t_{gM} \rangle$, we have

$$G_1 = C_{O(\Lambda)}(D_1) = C_{O(\Lambda)}(\langle u, t_M, t_{gM} \rangle) = C_{C_1}(u).$$

Next we study the action of $u$ on $(g^{-1}M) \cap J$. First we note that $u = t_E$ acts as $-1$ on $g^{-1}M \cap K \cong AA_2$ and $E \cap g^{-1}M \cong DD_4$. Thus $E \cap g^{-1}M \cap J \cong 2A_2$. Notice that $E \cap g^{-1}M \cap J$ is the $(-1)$-eigenlattice of $u$ in $g^{-1}M \cap J$ and $N \cap g^{-1}M$ is the fixed point sublattice of $u$ in $g^{-1}M \cap J$. Hence $u$ acts as $-t_{N \cap g^{-1}M}$ on $(g^{-1}M) \cap J$. Thus we have

$$C_{C_1}(u)/\langle -1, g_1 \rangle = C_{C_1}(t_{N \cap g^{-1}M})/\langle -1, g_1 \rangle$$

since $-1$ is in the center of $C_1$. Recall that $g_1$ is an order 3 isometry of $\Lambda$ as defined in (5) of (3.6), which acts fixed point free on $K$ and trivially on $J$. Note also that $C_1/\langle -1, g_1 \rangle \cong PSU(4,2).2 \cong Weyl(E_6)$ by (5). Since $t_{N \cap g^{-1}M}$ has trace $-2$ on $g^{-1}M \cap J$, by the character table of $PSU(4,2)$ [2, Page 26], we know that $C_{C_1}(u)/\langle -1, g_1 \rangle$ has the order $2 \times 576$ and has the shape $2.(Alt_4 \times Alt_4).2^2$.

Thus, $C_{O(\Lambda)}(\Delta)$ has the order $12 \cdot 576 = 2^9 \cdot 3^3$ and has the shape

$$(3 \times 2 \times 2.(Alt_4 \times Alt_4).2).2$$

as desired. □

By Lemma (2.11), Theorem 4.4 and Proposition 4.5, we deduce the main theorem of this section.

**Theorem 4.15.** *Let $(x, y, z)$ be a triple of elements in the Monster such that $x, y$ in $2A$, $xy$ in $6A$, $z \in 2B \cap C_{\mathbb{M}}(x, y)$ and $(xy)^3 \notin O_2(C_{\mathbb{M}}(z))$. Then such triples form one orbit under the conjugation action of $\mathbb{M}$ and $C_{\mathbb{M}}(x, y, z)$ has a homomorphism onto*

$$C_{O(\Lambda)}(\Delta)/\langle \pm 1\rangle \cong (3 \times 2.(Alt_4 \times Alt_4).2).2.$$

*The kernel $\tilde{K}$ is a group of order $2^9$ and the sequence*

$$1 \to \langle z\rangle \to \tilde{K} \to \{\varphi_\beta \mid \beta \in \Lambda \text{ and } \langle \beta, M + N\rangle \in 2\mathbb{Z}\} \to 1$$

*is exact.*

**Proof.** The kernel can be computed by the same argument as in Corollary (4.7). □

## 4.3 Isometry groups of $Q' \cong DIH_{12}(16)$ and $ann_\Lambda(Q')$

**Notation 4.16.** *Let $Q' = DIH_{12}(16)$ and $R' = ann_\Lambda(Q')$. Let $M$ and $N$ be $EE_8$ sublattices of $Q'$ such that $Q' = M + N$. As in Notation (4.10), we set $h = t_M t_N$, $g = h^2$ and $u = h^3$. Then $\langle u\rangle$ is the center of $\langle t_M, t_N\rangle$. Let $E$ be the $(-1)$-eigenlattice of $u$ in $Q'$. Then $E$ is also an $EE_8$-sublattice of $Q'$ and $E \cap M \cong E \cap N \cong DD_4$ (see [12]).*

*Set $F_M := ann_M(N)$ and $F_N := ann_N(M)$. Then by [12], we have $F_M \cong F_N \cong AA_2$ and $J := ann_{Q'}(F_M \perp F_N) \cong K_{12}$. In addition, $E \cap J \cong AA_2 \perp AA_2$.*

**Lemma 4.17.** *The isometry group $O(R') \cong O(A_2) \circ O(D_4)$ and has the order $2^8 \cdot 3^3$.*

*Proof.* Since $R' \cong A_2 \otimes D_4$ (A.1), we have $O(R') \cong O(A_2 \otimes D_4) \cong O(A_2) \circ O(D_4)$ by (3.27). □

**Proposition 4.18.** *Let $M, N$ be defined as in (4.16). Then the image of $\langle t_M, t_N\rangle$ in $O(Q')$ is normal in $O(Q')$.*

*Proof.* It follows from the classification of $EE_8$-sublattices in $Q'$ ( Lemma (A.6)). □

**Proposition 4.19.** *Let $y \in O(Q')$. Then $y$ normalizes the subgroup $\langle g \rangle$ and thus it stabilizes the sublattice $J = ann_{Q'}(Fix_{Q'}(g))$.*

*Proof.* Since $y$ normalizes $\langle t_M, t_N \rangle$ and $\langle g \rangle$ is the unique subgroup of order 3 in $\langle t_M, t_N \rangle \cong Dih_{12}$, $y$ also normalizes $\langle g \rangle$. □

**Remark 4.20.** *By the discussion in Case 2 of Appendix A, it is easy to see that $R'' := ann_{Q'}(E) = ann_J(E \cap J) \cong A_2 \otimes D_4 \cong R'$.*

**Theorem 4.21.** *Let $Q' \cong DIH_{12}(16)$ and $E$ be defined as in Notation (4.16). Let $R'' = ann_{Q'}(E)$ and let $\varphi' : O(Q') \to O(R'')$ be the restriction map. Then $Im(\varphi') = O(R'') \cong O(A_2) \circ O(D_4)$ and $Ker(\varphi') \cong 2.O^+(4,2)$. Therefore, $O(Q')$ has the shape $2.O^+(4,2).(O(A_2) \circ O(D_4))$.*

*Proof.* First we note that $O(Q')$ stabilizes the $EE_8$ sublattice $E$ and hence it also stabilizes $R'' = ann_{Q'}(E) \cong A_2 \otimes D_4$. Let $p_E : Q' \to \mathcal{D}(E)$ and $p_{R''} : Q' \to \mathcal{D}(R'')$ be the natural maps and let $\xi' : p_E(Q') \to p_{R''}(Q')$ be the gluing map from $E \perp R''$ to $Q'$. Then

$$O(Q') = \{y \in O(E) \times O(R'') \mid y\xi' = \xi' y\}.$$

By Lemma (4.14), we know that

$$R'' \cap N = R'' \cap g^{-1}M = N \cap g^{-1}M \cong DD_4$$

and $O(Q')$ acts as the full isometry group $O(D_4)$ on $R'' \cap N$. Similarly, we also have $R'' \cap g^i N (= R'' \cap g^{i-1}M) \cong DD_4$ and $O(Q')$ acts as $O(D_4)$ on each of $R'' \cap g^i N$ for $i = 0, 1, 2$. Moreover, the dihedral group $\langle t_N, t_M \rangle$ acts on $R''$ as $Dih_6$ with the kernel $\langle t_E \rangle$. More precisely, $\langle t_N, t_M \rangle$ acts as permutations on the set $\{R'' \cap M, R'' \cap gM, R'' \cap g^2 M\} = \{R'' \cap N, R'' \cap gN, R'' \cap g^2 N\}$. Hence, $O(Q')$ acts on $R''$ as the full isometry group $O(R'') \cong O(A_2) \circ O(D_4)$ and we have

$$Im(\varphi') = O(R'') \cong O(A_2) \circ O(D_4).$$

In this case, the kernel of $\varphi'$ is given by

$$Ker(\varphi') = \{y \in O(E) \mid y \text{ fixes } p_E(Q') \text{ pointwise } \}.$$

By our discussion in Case 2 of Appendix A, $[Q' : E \perp R''] = 2^4$ and $p_E(Q')$ forms a 4-dimensional non-degenerate quadratic spaces of $(+)$-type; in fact, it is isometric to the 2-part of $\mathcal{D}(AA_2 \perp AA_2)$ and is a direct sum of two non-singular 2-dimensional quadratic spaces of $(-)$-type. Recall that $O(E) \cong$

$2.O^+(8,2)$ and hence the subgroup $Ker(\varphi')$ that fixes $p'(Q')$ pointwise is, by Witt's theorem, isomorphic to $2.O^+(4,2)$. This group $Ker(\varphi')$ is actually isomorphic to a direct product $2 \times O^+(4,2)$ (for if $Y$ denotes the normal subgroup of $Ker(\varphi')$ generated by reflections on $R$, $Y \cong Dih_6 \times Dih_6$. Thus $Ker(\varphi')/Y$ has order 4. Now use (C.3)). $\square$

On the lattice $ann_{Q'}(J) \cong AA_2 \perp AA_2$, $t_M$ negates one of the $AA_2$ summands and fixes the other, while $t_N$ behaves analogously, negating the summand which $t_M$ fixes. It is clear from the analysis of $Ker(\varphi')$ that there exists an element of $Ker(\varphi')$ which interchanges the two $AA_2$ summands. Therefore, since $\langle t_M, t_N \rangle$ is normal in $O(Q')$, $O(Q')$ induces by conjugation the full automorphism group of $\langle t_M, t_N \rangle$ ($Aut(Dih_{12}) \cong Dih_{12}$).

**Corollary 4.22.** *$O(Q')$ leaves invariant the sublattice $E \perp R''$ and the restriction maps to $E, R'$ give an embedding of $O(Q')$ in $(2.[Dih_6 \times O^+(4,2)]) \times (Dih_6 \times O^+(4,3))$.*

**Remark 4.23.** *We have $O(D_4) \cong Weyl(F_4) \cong O^+(4,3)$.*

*The group $O(Q')$ acts on $\mathcal{D}(Q') \cong 6^4 \cong 2^4 \times 3^4$ as the full isometry group $O^+(4,2) \times O^+(4,3)$ and the kernel of the action is given by $\langle t_M, t_N \rangle \cong Dih_{12}$.*

# A Embeddings of $DIH_6(14)$ and $DIH_{12}(16)$ into the Leech lattice

Let $L \cong AA_2^{12}$ be the orthogonal sum of 12 copies of $AA_2$ and let $\mathcal{H}$ be the hexacode over $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ with the generating matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & \omega & 0 & \omega^2 & 0 & \omega^2 \end{pmatrix}$$

Let $\mathcal{D}$ be the ternary code generated by

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $\mathcal{D}$ is isomorphic to a sum of 3 ternary tetra-codes.

We can construct the Leech lattice by using $L$ and $\mathcal{D}$ and $\mathcal{H} \perp \mathcal{H}$ as glue codes [17].

**Case 1:** $DIH_6(14)$.

Let $X$ be the sum of the first 5 copies of $AA_2$. Then by the above construction,

$$Q := DIH_6(14) \cong ann_\Lambda(X).$$

Moreover,

$$R := ann_\Lambda(Q) = ann_\Lambda(S),$$

where $S$ is the sum of the last 7 copies of $AA_2$.

Note that $[R : AA_2^5] = 2^4$. In fact, one can glue $AA_2^5$ to $R$ by using the glue code generated by

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & \omega & \omega^2 & 0 & \omega^2 \end{pmatrix}$$

$$\begin{tikzcd}
 & \Lambda \arrow[dl] \arrow[dr] & \\
EE_8^3 \arrow[dr, "\mathcal{D}"'] & & J' \perp J \arrow[dl, "\mathcal{C}' \perp \mathcal{C}"] \\
 & AA_2^{12} &
\end{tikzcd}$$

**Case 2:** $DIH_{12}(16)$.

Let $Q' := DIH_{12}(16)$ and $R' := ann_\Lambda(Q')$. Then $\mathcal{D}(Q') \cong \mathcal{D}(R') \cong 6^4$.

Let $X'$ be the sum of the 1st, 2nd, 3rd and 4th copies of $AA_2$. Then

$$DIH_{12}(16) \cong ann_\Lambda(X').$$

By explicit calculation in the Leech lattice, one can show that $R'$ contains a sublattice isometric to $AA_2^{\perp 4}$ and $R'$ is isometric to

$$span_{\mathbb{Z}}\{AA_2^{\perp 4}, \frac{1}{2}(\beta_1, \beta_1, \beta_1, \beta_1), \frac{1}{2}(\beta_2, \beta_2, \beta_2, \beta_2)\},$$

where $\beta_1 = \sqrt{2}\alpha_1$, $\beta_2 = \sqrt{2}\alpha_2$ and $\{\alpha_1, \alpha_2\}$ is a set of fundamental roots for $A_2$. Note that $\mathcal{D}(AA_2) = 2 \cdot 6$.

By the analysis in [12], $Q'$ contains a sublattice

$$AA_2 \perp AA_2 \perp J,$$

where $J$ is isometric to the Coxeter-Todd lattice.

**Remark A.1.** *The lattice $R'$ is isometric to $A_2 \otimes D_4$.*

*Proof.* Recall that $R'$ is isometric to

$$span_{\mathbb{Z}}\{AA_2^{\perp 4}, \frac{1}{2}(\beta_1,\beta_1,\beta_1,\beta_1), \frac{1}{2}(\beta_2,\beta_2,\beta_2,\beta_2)\},$$

where $\beta_1 = \sqrt{2}\alpha_1$, $\beta_2 = \sqrt{2}\alpha_2$ and $\{\alpha_1,\alpha_2\}$ is a set of fundamental roots for $A_2$. Then the sublattices

$$D^1 = span\{(\beta_1,0,0,0),(0,\beta_1,0,0),(0,0,\beta_1,0),\frac{1}{2}(\beta_1,\beta_1,\beta_1,\beta_1)\} \quad \text{and}$$
$$D^2 = span\{(\beta_2,0,0,0),(0,\beta_2,0,0),(0,0,\beta_2,0),\frac{1}{2}(\beta_2,\beta_2,\beta_2,\beta_2)\}$$

are both isometric to $DD_4$ and $D^2 = g(D^1)$, where $g = (\sigma,\sigma,\sigma,\sigma)$ and $\sigma \in O(AA_2)$ is given by $\beta_1 \to \beta_2 \to -(\beta_1+\beta_2) \to \beta_1$. Then $D^1+D^2 \cong A_2 \otimes D_4$ by Lemma 3.1 [12]. Hence $R' \cong A_2 \otimes D_4$ since they have the same determinant. □

In this appendix, we shall determine all $EE_8$-sublattices in $Q' \cong DIH_{12}(16)$ and $Q \cong DIH_6(14)$.

**Lemma A.2.** *Let $J$ be isometric to the rank 12 Coxeter-Todd lattice $K_{12}$. Then $J$ contains no sublattices isometric to $EE_8$.*

**Proof.** Suppose $Y \cong EE_8$ is a sublattice of $J$. Then by Lemma A.3 of [12], the 3-rank of $\mathcal{D}(Y)$ is at least $6+8-12=2$. It is a contradiction since $\mathcal{D}(Y) \cong 2^8$. Note that the 3-rank of $\mathcal{D}(J)=6$, $rank(J)=12$ and $rank(E)=8$. □

**Lemma A.3.** *Let $A$ be a rank 2 even lattice with $\mathcal{D}(A) \cong 2^2 \cdot 3$ and minimal norm at least 4. Then $A \cong AA_2$.*

*Proof.* Let $\begin{pmatrix} 2a & c \\ c & 2b \end{pmatrix}$ be the Gram matrix of $A$. Then $4ab-c^2=12$ or $ab-(c/2)^2=3$. Hence, $c/2$ is an integer. Therefore, $A' = \frac{1}{\sqrt{2}}A$ is integral and $det(A')=3$.

Let $H(n,d) = (\frac{4}{3})^{(n-1)/2}d^{1/n}$ be the Hermite function (cf. [11, 16]). Then $H(2,3)=2.000$. Since $A$ has minimal norm at least 4, the minimal norm of $A'$ is at least 2. Thus $A'$ has a norm 2 vector $u$. Then $ann_{A'}(u)$ has determinant $\frac{1}{2}det(A')=3/2$ or $2det(A')=6$. Since $A'$ is integral, $ann_{A'}(u)$ has determinant 6. Let $v$ be a basis of $ann_{A'}(u)$. Then $w := \frac{1}{2}(u+v) \in A'$. Moreover, $w$ has norm $\frac{1}{4}(2+6)=2$ and $\langle u,-w\rangle = -1$. Thus $A' \cong A_2$ and $A \cong AA_2$. □

**Notation A.4.** *Let $Q' \cong DIH_{12}(16)$. Let $M$ and $N$ be $EE_8$ sublattices of $Q'$ such that $Q' = M + N$. In this case, the $EE_8$-involutions $t_M$, $t_N$ generate a dihedral group of order 12. As in Notation (4.10), we set $h = t_M t_N$, $g = h^2$ and $u = h^3$. Then $\langle u \rangle$ is the center of $\langle t_M, t_N \rangle$. Let $E$ be the $(-1)$-eigenlattice of $u$ in $Q'$. Then $E$ is also an $EE_8$-sublattice of $Q'$ and $E \cap M \cong E \cap N \cong DD_4$ (see [12]).*

*Set $F_M := ann_M(N)$ and $F_N := ann_N(M)$. Then by [12], we have $F_M \cong F_N \cong AA_2$ and $J := ann_{Q'}(F_M \perp F_N) \cong K_{12}$. In addition, $E \cap J \cong AA_2 \perp AA_2$.*

**Lemma A.5.** *Let $Y$ be an $EE_8$ sublattice of $Q'$. Then either $Y = E$ or $E \cap Y \cong DD_4$.*

*Proof.* Suppose $Y \neq E$. Then by the classification of $EE_8$-pairs [12], we have $E \cap Y \cong 0, AA_1$, $AA_1^2$, $AA_2$ or $DD_4$.

**Case 1:** $E \cap Y = 0$. In this case, $E + Y$ is a full rank sublattice of $Q'$. By the classification of $EE_8$-pairs and $\mathcal{D}(Q') = 6^4$, the only possible case is $E + Y = Q'$. Then we have $F_E := ann_E(Y) \cong AA_2$, $F_Y := ann_Y(E) \cong AA_2$ and $ann_E(F_E) \cong EE_6$. Note also that $F_Y \subset ann_{Q'}(E) \cong R'$ and $ann_{R'}(AA_2) \cong AA_2^3$. Thus, we obtain a full sublattice of type $EE_6 \perp AA_2^3$ in $ann_{Q'}(F_E \perp F_Y) \cong K_{12}$. It is impossible since the 3-rank of $\mathcal{D}(EE_6 \perp AA_2^3)$ is 4 and $\mathcal{D}(K_{12}) = 3^6$.

**Case 2:** $E \cap Y \cong AA_1$. Then $E + Y \cong DIH_8(15)$. In this case, the $EE_8$ involutions $t_E, t_Y$ generate a dihedral group of order 8. Let $z = (t_E t_Y)^2$ and let $Z$ be the $(-1)$-eigenlattice of $z$ in $Q'$. Let $F_E$ and $F_Y$ be the fixed point sublattices of $z$ in $E$ and $Y$, respectively. By the analysis in [12], we know that $F_E \cap F_Y \cong AA_1$, $F_E \cong F_Y \cong DD_4$ and $Z \cong EE_8$. Then $ann_{E+Y}(F_E \perp Z) \supset ann_{F_Y}(F_E \cap F_Y) \cong AA_1^3$. However, $ann_{Q'}(F_E \perp Z) \cong \sqrt{6}D_4$ has no norm 4 vectors. It is a contradiction.

**Case 3:** $E \cap Y \cong AA_1^2$. Then $R' \supset ann_Y(E) \cong DD_6$ but $R' \cong A_2 \otimes D_4$ doesn't have such a sublattice.

Recall that the norm 4 vectors of $R'$ have the form $\alpha \otimes \beta$, where $\alpha, \beta$ are roots of $A_2$ and $D_4$, respectively. Suppose $R'$ contains a sublattice isometric to $DD_6$. Since $DD_6 \supset AA_1^6$, there exist roots $\alpha_1, \dots, \alpha_6 \in A_2$ and $\beta_1, \dots, \beta_6 \in D_4$ such that

$$\langle \alpha_i \otimes \beta_i, \alpha_j \otimes \beta_j \rangle = \langle \alpha_i, \alpha_j \rangle \cdot \langle \beta_i, \beta_j \rangle = 4\delta_{i,j},$$

for any $i, j = 1, \dots, 6$. Since $\langle \alpha_i, \alpha_j \rangle \neq 0$ for any roots $\alpha_i, \alpha_j \in A_2$, we must have $\langle \beta_i, \beta_j \rangle = 2\delta_{i,j}$ for $i, j = 1, \dots, 6$. It is impossible because $D_4$ has rank 4.

**Case 4:** $E \cap Y \cong AA_2$. Then $E+Y \cong DIH_6(14)$ and $ann_{E+Y}(E \cap Y) \cong K_{12}$. By the analysis in [12], we have

$$ann_E(E \cap Y) \cong EE_6 \quad \text{and} \quad (E+Y) \cap R' = ann_{E+Y}(E) \cong \sqrt{6}E_6^*. \quad (6)$$

Let $A = ann_{Q'}(E+Y)$. Then $\mathcal{D}(A) \cong 2^2 \cdot 3$. Then $A \cong AA_2$ by Lemma A.3. Then $ann_{R'}(A) \cong AA_2^3$. However, $ann_{R'}(A) = (E+Y) \cap R' \cong \sqrt{6}E_6^*$ by (6). It is a contradiction.

Therefore, the only possible case is $E \cap Y \cong DD_4$. Note that such a case occurs since $E \cap M \cong E \cap N \cong DD_4$ (see [12]). □

**Proposition A.6.** *We use the same notation as in Notation (4.16). Let $Y$ be an $EE_8$-sublattice in $Q'$. Then $Y = E$, $g^i(M)$ or $g^i(N)$, where $g = (t_M t_N)^2$ and $i = 0, 1, 2$.*

*Proof.* Let $Y$ be an $EE_8$ sublattice in $Q'$ and $E \neq Y$. Then $E \cap Y \cong DD_4$ and $E+Y \cong DIH_4(12)$. Moreover, we have $Y \cap R' = ann_Y(Y \cap E) \cong DD_4$. Since there are only 3 $DD_4$-sublattices in $R'$, we must have $Y \cap R' = g^i(M) \cap R'$ for some $i = 0, 1, 2$. It now follows $E + Y = E + g^i(M)$ and we have the desired conclusion since $DIH_4(12)$ has only 3 $EE_8$ sublattices in it. □

**Proposition A.7.** *Let $Q \cong DIH_6(14)$. Then $Q$ contains exactly three $EE_8$-sublattices.*

*Proof.* Let $M$ and $g$ be defined as in Notation A.4. Then $M + gM \cong DIH_6(14)$. Thus we can view $Q \cong DIH_6(14)$ as a sublattice of $Q'$. The result now follows from Proposition (A.6). □

# B The containment $C_{O(\Lambda)}(\langle t_M, t_N \rangle)$ in $C_{O(\Lambda)}(t_E)$

The group $C_{O(\Lambda)}(t_E)$ has the form $2.BRW^+(2^4) \cong (2 \times 2^{1+8})\Omega^+(8,2)$. The groups $BRW^+(2^d)$ may be analyzed by the methods of [11]. We give some discussion of $C_{O(\Lambda)}(t_E)$ and $C_{O(\Lambda)}(\langle t_M, t_N \rangle)$ from the BRW-viewpoint.

**Lemma B.1.** *Let $u$ and $g$ be defined as in (4.16). The element $g \in C_{O(\Lambda)}(u)$ acts on the Frattini quotient of $O_2(C_{O(\Lambda)}(u))$ with 4-dimensional fixed points. In the notation of [9], $g$ is in the class of $\theta_2$.*

**Proof.** An element of order 3 in the class of $\theta_i$ acts on $Fix(u)$ with trace $-8, 4, -2, 1$ for $i = 1, 2, 3, 4$, respectively.

By (4.10), the sum of the traces in $\langle g, u\rangle$ is 48. By the orthogonality relations, $rank(Fix(g) \cap \Lambda^+(u)) = 8$ whence $rank(Fix(g) \cap \Lambda^-(u)) = 4$. This implies that the trace of $g$ on $\Lambda^+(u)$ is $8 - \frac{1}{2}(16 - 8) = 4$, whence $i = 2$, as claimed. □

**Lemma B.2.** *Let $t \in \{t_M, t_N\}$. Then $t$ acts with trace 8 on $\Lambda^+(u)$ and trace 0 on $\Lambda^-(u)$.*

**Proof.** Define $U := O_2(C_{O(\Lambda)}(u))$. The trace of $t$ on $\Lambda^+(u)$ is 0 if and only if $t$ is conjugate in $C_{O(\Lambda)}(u)$ to $tu$ and otherwise the trace has the form $\pm 2^d$ where $2d$ is the dimension of the fixed points of $t$ in its action on $U/U'$.

We claim that $t$ does not have trace 0. Supposing otherwise, we see that $t$ has trace 8 on $\Lambda^-(u)$, which means that $t \in U$. This is impossible since $t$ inverts $g$. The claim follows.

Suppose $t = t_M$. The rank of $M \cap E$ is 4 by (4.12). Therefore, $\Lambda^+(u) \cap M$ has rank 4. This leads to trace $12 - 4 = 8$ on $\Lambda^+(u)$. Trace 0 on $\Lambda^-(u) = E$ follows. □

**Lemma B.3.** $Weyl(F_4) \cong O(D_4) \cong O^+(4, 3)$.

**Proof.** The group $Weyl(F_4)$ acts faithfully as isometries on the $D_4$-lattice, which is spanned by the long roots of $F_4$. This lattice has determinant 4, whence its reduction modulo 3 is nonsingular. The $\mathbb{F}_3$-valued form is split since the sublattice of type $A_1^4$ maps onto $D_4/3D_4$ (take an orthogonal set of roots $p, q, r, s$ and note that $p + q + r, q - r + s$ generates a direct summand of $\mathbb{Z}p + \mathbb{Z}q + \mathbb{Z}r + \mathbb{Z}s$ and that their image in $D_4/3D_4$ is totally singular). We therefore get a homomorphism $Weyl(F_4) \to O^+(4, 3)$. Both groups have order $2^7 3^2$ and this map is monic because any normal subgroup of $Weyl(F_4)$ contains the central involution which acts as $-1$ on $D_4/3D_4$. □

**Notation B.4.** *Let $G := BRW^+(2^4)$, $U := O_2(G) \cong 2_+^{1+8}$, $\tau \in G$ an involution of trace 8 on the natural module (dimension 16) and $\theta \in G$ an element of order 3 such that $[U, \theta]/U'$ has rank 4. Define $G_1 := C_G([U, \theta])$, $G_2 := C_G(G_1)$. Then $G_1 \cong G_2 \cong BRW^+(2^2)$.*

**Lemma B.5.** $[U, \tau] \leq [U, \theta]$.

**Proof.** Since the trace of $\tau$ is 8, $[U, \tau]/U'$ has rank 2. Since $\tau$ inverts $\theta$ by conjugation and leaves invariant $[U, \theta]$, the containment follows. □

**Corollary B.6.** *$\tau \in G_2$.*

**Proof.** Note that the direct product decomposition $[U,\theta]/U' \times C_U(\theta)/U'$ is orthogonal in the sense of the natural quadratic form on $U/U'$. The stabilizer of one summand stabilizes both and that stabilizer is $G_1G_2$, a central product. So $\tau \in G_1G_2$ and since $\tau$ stabilizes both summands and has commutator rank 2 on the quadratic space, $\tau \in UG_2$. If $\tau \notin G_2$, there exists $x \in (U \cap G_1) \cap G_2 \setminus U'$ so that $\tau \in xG_2$. There exists $y \in U \cap G_1$ so that $[x,y]$ generates $U'$. Then $[y,\tau]$ generates $U'$. This means that $\tau$ has trace 0 on the natural module, a contradiction. □

The next two results apply to $\theta, \tau \in G_2 \cong Weyl(F_4)$ acting on the $D_4$-lattice.

**Lemma B.7.** *In $O(D_4)$, if $P \cong 3^2$ is a Sylow 3-group and $H$ is its normalizer, $H = Z(O_2(O(D_4))) \times H_1 \times H_2$, for a unique pair of dihedral subgroups $H_1, H_2$ generated by reflections. Furthermore, an element of order 3 in an $H_i$ has trivial fixed points on the Frattini quotient of $O_2(O(D_4))$. Elements of order 3 in $P \setminus (H_1 \cup H_2)$ have rank 2 fixed points on the $D_4$-lattice.*

**Proof.** We observe that $Weyl(F_4)$ contains a natural $Weyl(A_2) \times Weyl(A_2)$ generated by reflections (at roots of different lengths). Since $P$, a Sylow 3-group of this (also a Sylow 3-group of $O(D_4)$), acts without fixed points on the Frattini factor of $O_2(O(D_4))$, $H \cong Z(O_2(O(D_4))) \times Dih_6 \times Dih_6$. The subgroup of this generated by roots is the group $Weyl(A_2) \times Weyl(A_2)$ mentioned above.

For the second, let $x$ be an element of order 3 in $H_1$. Since $x$ has trace 1 on the lattice, $x$ has trivial fixed points on the Frattini quotient of $O_2(O(D_4))$. The remaining four elements of order 3 in $P$ have rank 2 fixed points on the Frattini factor and 0 fixed point sublattice.

Under conjugacy by $H$, the elements of order 3 are partitioned into orbits $\{x, x^{-1}\}$, $\{y, y^{-1}\}$ and the remaining set of four elements of order 3 (the ones which have nontrivial fixed points on the Frattini quotient of $O_2(O(D_4))$. The final statement follows. □

**Corollary B.8.** *We use the notation of (B.7). Let $\{i,j\} = \{1,2\}$. If $\theta \in H_i$, then $C_{O(D_4)}(\theta) = Z(O_2(O(D_4))) \times \langle\theta\rangle \times H_j$. Also, $\tau \in H_i$ and $C_{O(D_4)}(\langle\theta,\tau\rangle) = Z(O_2(O(D_4)))H_j$.*

**Proof.** Since $\theta$ centralizes just $Z(O_2(O(D_4)))$ on $O_2(O(D_4))$, the form of $C(\theta)$ follows from (B.7). An element of $H$ which inverts $P$ has trace 0 on

the lattice since it induces outer automorphisms on each quaternion group in $O_2(O(D_4))$. Therefore, $\tau \in Z(O_2(O(D_4)))r$, where $r \in H_i$ is a reflection. Since $r$ has trace 2 and since the central involution acts as $-1$ on the lattice, $\tau$ has trace $\pm 2$. Since $\tau$ has trace 8 on the rank 16 representation (B.4) and this module is a tensor product for the central product $G_1 G_2$, it follows that in the action of $G_2$ on $D_4$, $\tau$ has trace 2 and $\tau$ is in fact a reflection. $\square$

**Notation B.9.** *Let $t_M, t_N, t_E, h, g$ be as in Section (on $DIH_{12}(16)$). Then $u := t_E$ is the central involution of $\langle t_M, t_N \rangle$.*

*Let $\psi$ be a homomorphism of $C_{O(\Lambda)}(u)$ onto $BRW^+(2^4)$. Denote $\tau := \psi(t_M), \theta := \psi(g)$. Note that (B.1), (B.2) imply that Notation (B.4) applies to $\tau, \theta$.*

*The centralizer in $G := BRW^+(2^4)$ of $\langle \tau, \theta \rangle$ has been discussed in (B.8). Note that $Ker(\psi) = \langle -u \rangle$ and that $\psi$ maps $C_{O(\Lambda)}(\langle t_M, t_N \rangle)$ to a subgroup of $C_G(\langle \tau, \theta \rangle) \cong O^+(4,3) \times Dih_6$. According to (4.14), $C_{O(\Lambda)}(\langle t_M, t_N \rangle)$ has order $2^8 3^3$, the same as $O^+(4,3) \times Dih_6$. This means that $\psi(C_{O(\Lambda)}(\langle t_M, t_N \rangle))$ has index 2 in $C_G(\langle \tau, \theta \rangle) \cong O^+(4,3) \times Dih_6$.*

# C Trivial action on lattices mod 2

**Lemma C.1.** *Suppose that the involution $t$ acts on the abelian group $L$ which has no 2-torsion. Assume that $t$ is trivial on $L/2L$. Then $L$ is the direct sum of eigenlattices for $t$.*

**Proof.** There exists an endomorphism $a$ of $L$ so that $t = 1 - 2a$. Then $1 = t^2 = (1-2a)^2 = 1 - 4a + 4a^2 = 1 + 4(a^2 - a)$ and absence of 2-torsion imply that $0 = a^2 - a = a(a-1)$.

For $x \in L$, $x = ax + (1-a)x$, whence $L$ is the sum of subgroups $Ker(1-a)$ and $Ker(a)$.

Let $x \in Ker(a)$. Then $tx = (1-2a)x = x$. If $y \in Ker(1-a)$, $ty = (1-2a)y = (-1 + 2(1-a))y = -y$. Therefore $L$ is the sum of the 1-eigenlattice and $(-1)$-eigenlattice of $t$. Their intersection is 0 since $L$ is free of 2-torsion. $\square$

**Corollary C.2.** *Suppose that the involution $t$ is an isometry of the orthogonally indecomposable rational lattice $L$ such that $t$ acts trivially on $L/2L$. Then $t = 1$ or $t = -1$.*

**Proof.** By the Lemma, $L$ is a direct sum of eigenlattices for $t$. Since $t$ is an isometry of $L$, this is an orthogonal direct sum, whence indecomposability of $L$ implies that one of the summands is 0. $\square$

**Lemma C.3.** *Let $L$ be a finite rank positive definite orthogonally indecomposable lattice. Suppose that $u \in O(L)$ has the property that $u^2$ acts trivially on $L/2L$ and $(u-1)(L/2L)$ has dimension less than $\frac{1}{2}rank(L)$. Then $u$ is an involution.*

**Proof.** Since $L$ is an orthogonally indecomposable lattice, $u^2 = \pm 1$ by (C.2). Suppose that $u^2 = -1$. Then $L$ is a torsion free module for $\mathbb{Z}[u] \cong \mathbb{Z}[\sqrt{-1}]$, a PID. Therefore, $L$ is a free module and $L/(u-1)L$ has $\mathbb{F}_2$-dimension exactly $\frac{1}{2}rank(L)$, a contradiction. $\square$

# D Facts about discriminant groups

**Lemma D.1** (Lemma A.9 of [14]). *Let $X$, $Y$ be sublattices of the lattice $L$ where $Y$ is a direct summand of $L$, $L = X + Y$ and $(det(Y), det(L)) = 1$. Then the natural map of $X$ to the discriminant group $\mathcal{D}(Y) = Y^*/Y$ is onto.*

**Lemma D.2.** *Let $X$ and $Y$ be sublattices of the unimodular lattice $L$ such that $X \cap Y = 0$ and $X + Y$ is a direct summand of $L$. Then for any $y \in Y^*$, there exists $\beta \in L$ such that (1) $P_Y(\beta) = y \mod 2Y$ and (2) $\langle P_X(\beta), X \leq 2\mathbb{Z}$.*

**Proof.** Since $X \cap Y = 0$, Conditions (1) and (2) define uniquely a $\mathbb{Z}$-linear map $\varphi : X+Y \to \mathbb{Z}_2$ such that $\varphi(\alpha) = \langle y, \alpha\rangle \mod 2$ for $\alpha \in Y$ and $\varphi(\alpha) = 0$ for $\alpha \in X$. Moreover, $X + Y$ is a direct summand of $L$ and thus the natural map from $L$ to $(X + Y)^*$ is a surjection. Hence there is a $\beta \in L$ such that $\varphi(\alpha) = \langle \beta, \alpha\rangle \mod 2$ for all $\alpha \in X + Y$ and $\beta$ satisfies Conditions (1) and (2). $\square$

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